Liquidity Stock and Bank Lending

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Abstract

Basel III requires banks to hold adequate stocks of liquid assets. This paper shows that a bank's liquidity stock has a profound effect on its cost of obtaining liquidity under information asymmetry, thereby having a complicated relationship with its lending behaviour. The lending rate is the lowest both when the bank's liquid assets are scarce and when they abound; it discontinuously surges when their size ascends above a threshold; it varies non-monotonically with the size. This is driven by a new effect. Because liquid assets are also safe assets, exchanging risky assets for liquidity amounts to inverse risk-shifting, which reduces the equity value and countervails lemons-dumping incentives. If the liquid stock is sufficiently low, this inverse risk-shifting effect is strong enough to overcome the lemons problem, which, consequently, incurs no costs, as is in the case where the liquidity stock is sufficiently large to cover the bank's liquidity needs so no external liquid is sought, nor the lemons problem present.

Key words: Liquidity Coverage Ratio, liquidity stock, bank lending, the lemons problem, inverse risk-shifting

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1 Introduction

In response to the 2008 financial crisis, the Basel Committee on Banking Supervision introduces the Liquidity Coverage Ratio standard. It requires a bank to hold an adequate stock of High Quality Liquid Assets "to meet its liquidity needs for a 30 calendar day liquidity stress scenario."[1] This new standard is meant to improve the resilience of banks in stressful times. But how would it affect banks' lending behaviour in normal times? One might intuitively think that a larger liquidity stock, by giving the bank a greater capacity to meet its liquidity demands, should always be a good thing, leading to a lower lending cost and rate. However, negative effects are suggested by Malherbe (2014), when the market for banks to obtain external liquidity is afflicted by the lemons problem. In this paper, we show that a bank's liquidity stock has much more profound, complicated effects on its lending behaviour than suggested by both of them. To a large extent, this is driven by the interaction between a bank's cost of obtaining liquidity with its endogenous leverage ratio, which leads to a new effect that has eluded the existing relevant studies.

The model economy lasts for three dates, as is typically with a model on bank liquidity. At date 0, one bank is endowed a stock of liquid assets and lend to a continnum of entrepreneurs. Their demand for loans is a decreasing function of the lending rate. The lending rate therefore determines the lending scale. The bank finances loans by issuing demand deposit contracts. The lending scale therefore pins down the deposit scale. The deposit contract gives the depositors the right to withdraw its full present value at time 1. At date 1, a fraction of depositors demands withdrawals. If the bank's lending scale is above a threshold, its liquidity stock is insufficient to meet the withdrawal demands and the bank needs to sell loans for liquidity (or alternatively, borrowing liquidity colllateralised with loans). This exchange is beset by the classic lemons problem of Akerlof (1970): At date 1 the bank has private information about the quality of loans and its exchanging loans for liquidity might be not because of its liquidity needs, but because it knows the loans are of low quality – i.e. lemons – and wants to dump them. The lemons problem tends to increase the cost of the bank obtaining external liquidity at date 1. Thereby, it affects the bank's lending decision at time 0, *unless* the bank elects its lending scale to be within an upper bound so that it resorts to no external liqudity, but uses its liquidity stock only, to meet all the withdrawal demands. The tightness of this scale constraint depends the size of its liquidity stock. If the stock is above a high threshold, the scale constraint is non-binding, the date-1 lemons problem incurs no cost, and the bank's lending rate is at the lowest level (its scale the highest).

Interestingly, so is it rate if the bank's liquidity stock is meagre enough. This is driven by the aforementioned new effect. Observe that liquid assets are typically also safe assets and thereby assume a *double*

[1] See "Basel III: The Liquidity Coverage Ratio and liquidity risk monitoring tools", page 10. https://www.bis.org/publ/bcbs238.htm.

identity. Loans, as having uncertain quality, are risky assets. Exchanging loans for liquidity, therefore, amounts to a swap of risky assets with the risk free one, which is the opposite of classic risk-shifting à la Jensen and Meckling (1976), or *inverse risk-shifting.* This swap, as is well known, increases the debt's value and decreases the equity's *if the debt is risky.* The loss to the bank engenders disincentives from dumping lemons. Moreover, the inverse risk-shifting effect is stronger if the liquidity stock is smaller. Indeed, the loss to the equity from inverse risk-shifting is the flip side of the gain from risk-shifting, and hence, is larger if the debt suffers a greater loss in the event of default, which happens if the liquidity stock, as a safe asset, is smaller. Hence, if the liquidity stock is sufficiently small, below a low threshold, the inverse risk-shifting effect is strong enough to overcome the lemons problem: Regardless of the loan quality, the bank exchanges for the same quantity of liquidity, the one exactly sufficient to meet the withdrawal demands, as if it had no private information. As a result, if the liquidity stock is below the *low* threshold, the date 1 lemons problem incurs no costs and the lending rate is at the lowest level, as in the case where the liquidity stock is above the *high* threshold.

Between these two ends, the relationship of the liquidity stock to the lening rate is non-monotonic and discontinuous. (a) The discontinuity. If the liquidity stock goes above the low threshold, the inverse risk-shifting effect is not strong enough to overcome the lemons problem. Consequently, the bank dumps lemons, as much as possible. This rationally expected, the cost of external liquidity surges *discontinuously* at the threshold, and so does the lending rate. (b) The non-monotonicity. First, if the liquidity stock is below but close to the high threshold, the bank still elects to subject itself to the scale constraint, but now the constraint is binding, though not that tight. Hence, the lending scale is at the upper bound where the liquidity stock *exactly* covers the withdrawal demands. In this scenario, the lending scale increases, and hence the lending rate *decreases*, with the liquidity stock. Intuitively, this conforms with the intuitive thinking: The higher the stock, the greater the capacity to absorb losses and meet liquidity needs, which should allows for a larger lending scale and a smaller lending rate. Second, if the liquidity stock is in another interval, the lending rate *increases* with the liquidity stock. The mechanism is the one found by Malherbe (2014)[2]: The higher the liquidity stock, the less likely is the bank seeking external liquidity because of genuine liquidity needs, the more likely for dumping lemons. As a result, the external liquidity is more costly, the lending rate higher.

This paper demonstrates that a bank's liquidity stock has a profound impact on its lending behaviour when the market for external liquity is beset by the lemons problem. At the core is the inverse risk-shifting effect, resulting from the interaction between the bank's liquidity stock (as safe assets) and its endogenous leverage. A large strand of literature, following the seminal work of Diamond and Dybvig (1983), consider

[2] Malherbe (2014), however, is not concerned with the relationship of a bank's liquidity stock with its lending rate and scale. Moreover, his paper focuses on the allocation of a given quantity of funds and is unconcerned with leverage on the liability side, whereas the bank's liability-side decision plays a key part in the present paper.

bank liquidity from the angle of depositors sharing liquidity risks, with which we are not concerned. Our concern is how a bank's stock of liquid assets affects its lending scale and rate, a subject with which the literature is uninterested. Closer to our paper are the studies that examine how banks' ex ante investment decisions are affected by the lemons problem on the liquidity market; see Bolton et al (2011), Heider et al (2015), Kirabaeva (2011), Malherbe (2014) and Parlour and Plantin (2008).[3] While the first four studies show that the lemons problem, by reducing the liquidity of the resale market, leads to inefficient capital allocation, market breakdowns, and multiple equilibria, Parlour and Plantin (2008) show that the illiquid secondary market can improve efficiency by preserving the bank's ex ante incentive of monitoring. Those studies are all abstracted from the liability-side contracts of the banks. In contrast, we consider the liability-side contracting problem and thereby discover the inverse risk-shifting effect, which countervails the lemons problem. Moreover, all those papers are abstracted from either the lending rate or the liquidity holding and therefore unconcerned with the effects of the liquidity stock on the lending rate, with which this paper is interested.

In this paper, if the bank's optimal leverage ratio is above a threshold, the inverse risk-shifting effect is strong enough to overcome the lemons problem. In a similar line, Bond and Leitner (2015) consider an interaction of the lemons problem with the leverage of the *buyer* of the lemon assets, whereas the seller' leverage we consider. Their interest is on market freeze and liquidity dry, also different from ours. Lastly, they are not concerned with the liquidity management, namely the satisfaction of liquidity needs in the interim.[4] On the other hand, Gomez and Vo (2020) examine the implications of banks' leverage for the liquidity management, but are not concerned with the lemons problem.

In this paper, the bank lends out its liability − a promise to pay − to entrepreneurs who then use it as a means of wage payment to hire workers and these workers thus hold the bank's liability. This way of modelling bank lending in a general-equilibrium framework is revived by a recent strand of literature; see among others Bianchi and Bigio (2017), Donaldson et al (2018), Jakab and Kumhof (2015), Mendizábal (forthcoming), Morrison and Wang (2018), and Wang (2019, 2021). In particular, Morrison and Wang (2018) examine the effect of banks' liquidity stock on their lending behaviour in a setting where depositors withdraw only when they are worried about the default risks of their banks. They find that the effect is nil if depositors have homogeneous information on the asset quality of the banks; and is always positive if they have heterogeneous information. In this paper, by contrast, depositor can also withdraw out of reasons unrelated to the bank's default risk, and the liquidity stock has a non-monotonic, discontinuous effect on

[3] While these studies, including the present one, hinge on the effect of the lemons problem on trading, Kurlat (2018) studies the effects of trading on the dynamics of the lemons problem.

[4] Similarly, Bigio (2015), Eisfeldt (2004) and Kurlat (2013) consider the dynamics of the lemons problem, but are not concerned with banks' liquidity management.

the bank's lending behaviour. Bianchi and Bigio (2017) and Wang (2021) examine the implication of banks' liquidity management for monetary policy in models where liquidity borrowing is not beset by the lemons problem.

2 The Model

The economy lasts for three dates: $t \in \{0,1,2\}$. It is populated by four types of risk-neutral agents: one bank, a continuum $[0,1]$ of entrepreneurs, a lot more workers, and a large number N of investors. In a nutshell, entrepreneurs borrow the bank's demandable liability as a means of payment to hire workers at $t=0$; the bank faces liquidity demand at $t=1$, to meet which it might need obtain liquidity supplied by the investors; and contractual obligations are settled at $t=2$. There is one consumption good, corn, which is storable. Corn also represents the liquid asset in the model economy. At date 0, the bank is endowed with G units of corn, the investors each with one unit.

At date 0, workers work either for entrepreneurs or in autarky. In autarky, a worker produce one unit of corn at date 2. If an entrepreneur employs L workers, then she produces the following quantity of corn at date 2:

$$y = \widetilde{A} K^{1-\alpha} L^{\alpha},$$

where K is the quantity of her own capital, $\alpha \in (0,1)$ and $\widetilde{A}$ represents the productivity shock that realises at date 2 and has the following distribution at date 0.

$$\widetilde{A} = \begin{cases} \overline{A}, & \text{with probability } \widetilde{q}; \\ \underline{A}, & \text{with probability } 1-\widetilde{q}, \end{cases} \tag{1}$$

where $0 < \underline{A} < \overline{A}$ and $\widetilde{q}$ is the common quality shock. At date 0,

$$\widetilde{q} = \begin{cases} q_L, & \text{with probability } p > 0; \\ q_H, & \text{with probability } 1-p > 0, \end{cases} \tag{2}$$

where $0 < q_L < q_H < 1$. The shock $\widetilde{q}$ realises at date 1. Its mean value is denoted by

$$q_e := p q_L + (1-p)\, q_H.$$

Without loss of generality, we normalize $K=1$. We assume that there are more workers than can be employed by entrepreneurs. As a result, an entrepreneur can hire workers by paying them with what they would earn in autarky, that is, one unit of corn.

Entrepreneurs have no corn at date 0. They would want to hire workers by promising to pay them at date 2 with the corn produced then. However, we assume that entrepreneurs have inadequate commitment power so that their promise to pay is not trusted by workers. By contrast, the bank have adequate commitment power so that workers accept its promise to pay as a means of wage payment. Put differently, the following is assumed.

Assumption 1: At date 0, workers supply labour in exchange for the bank's promise to pay, but not for entrepreneurs'.

This assumption captures the observation that in reality, banks' liabilities such as demand deposits are widely accepted as a means of payment, whereas non-banks' are not. With the assumption, the bank's role at $t = 0$ is to lend its liability to entrepreneurs who use it as a means of wage payment to obtain labour from workers. Eventually, entrepreneurs owe a debt to the bank, the bank to the workers. This is equivalent to the arrangement in which workers "deposit" their labour with the bank and the bank then lends it out to entrepreneurs, if this arrangement is imaginable. In this arragement, the intermedation of the bank is necessary for entrepreneurs to obtain labour from the workers, again, because of Assumption 1.

As a result of Assumption 1, at date 0 entrepreneurs need borrow the bank's promise to pay in order to hire workers. A liability contract is represented by (r_1, r_2), by which the bank promises to pay r_1 at date 1 *or* r_2 at date 2 and the contract bearer decides which date to demand the payment. We define *one unit of bank liability* as one contract (r_1, r_2) with which an entrepreneur hires one work. The market value of one unit of bank liability is thus equal to the wage per worker, namely, one unit of corn.

Equivalently, the lending of the bank's liability is conducted via textbook loan-deposit-loan cycles: First entrepreneurs borrow corn from the bank to hire workers (who certainly accept corn as their wage payment), using one unit of corn to hire one worker; the workers deposit the corn that they receive as the wage payment with the bank, and the deposit contract is (r_1, r_2) for one unit of corn; and the bank lends corn out again. With this interpretation, (r_1, r_2) is a *deposit* contract, and the contract bearers are *depositors*; and one unit of the deposit contract is used by the bank to exchange one worker's wage, that is, one unit of corn.

Demanding payment at date 1, therefore, amounts to early withdrawal. We assume that deposit contract that workers accept as a means of wage payment is *fully demandable*, that is, it entitles the depositor to withdraw its full value at date 1. This assumption is justified by the observation that in reality the most common form of bank liability that is widely used as a means of payment is demand deposit, which entitles the depositor to fully withdraw his claim on demand.[5] By this assumption, one unit of bank liability gives the depositor rights to withdraw one unit corn at date 1 and is thus represented by $(1, r_2)$. This r_2 is the gross deposit rate between dates 1 and 2.

[5] In this paper, we do not derive demandability as a part of equilibrium; for studies that derive it, see Diamond and Dybvig (1983) and Morrison and Wang (2019).

The relationship between the bank and an entrepreneur is governed by lending contract (D, R), by which the entrepreneur acquires D units of bank liability at date 0 and is obligated to pay the bank DR units of corn at date 2; thus R is the gross lending rate. We define *one unit of loan* as a loan of unit face value. Then, by entering DR units of loans, an entrepreneur acquires D units of liability, whereby she hires D workers and produces $\widetilde{A}D^{\alpha}$ units of corn at date 2. The decision problem of entrepreneurs at date 0 is as follows.

$$\max_{D} q_e \left(\overline{A}D^{\alpha} - DR\right) + (1 - q_e) \max\left(\underline{A}D^{\alpha} - DR, 0\right), \tag{3}$$

where the "max" term takes care of the possibility of entrepreneurs default in the bad state.

We assume the following inequality holds:

$$\frac{\underline{A}}{\overline{A}\alpha} < \frac{q_e}{1 - \alpha (1 - q_e)}. \tag{4}$$

As a result,

$$\frac{\underline{A}}{\overline{A}\alpha} < 1$$

because $q_e / (1 - \alpha (1 - q_e)) < q_e / (1 - (1 - q_e)) = 1$. Condition (4) means that the bad-state productivity $\underline{A}$ is low enough, hence the following lemma.

Lemma 1 *At the optimum, entrepreneurs default in the bad state and repays each unit of loan with $\frac{\underline{A}}{\overline{A}\alpha}$ unit of corn. At date 0, their demand of bank liability depends on the lending rate R as follows:*

$$D = \left(\frac{\overline{A}\alpha}{R}\right)^{\frac{1}{1-\alpha}} := D(R). \tag{5}$$

Proof. See Appendix A. ■

According to Lemma 1, one unit of loan receives a repayment of one unit of corn in the good state and $\underline{A}/\left(\overline{A}\alpha\right)$ unit in the bad state. If the good state occurs with probability $\widetilde{q}$, the ex ante value of a unit of loan is

$$\delta(\widetilde{q}) = \widetilde{q} \times 1 + (1 - \widetilde{q}) \frac{\underline{A}}{\overline{A}\alpha}; \tag{6}$$

and the market value of any loan is equal to the product of its face value and the discount factor $\delta(\widetilde{q})$. The probability $\widetilde{q}$ measures the quality of the loans. If $\widetilde{q} = q_L$, loans are of low quality, discounted by

$$\delta_L := \delta(q_L).$$

If $\widetilde{q} = q_H$, they are of high quality, discounted by

$$\delta_H := \delta(q_H).$$

We let

$$\delta_e := p\delta_L + (1 - p)\delta_H = \delta(q_e). \tag{7}$$

At date 1, depositors might exercise their withdrawal right for a variety of reasons. First, a fraction $\omega < 1$ of depositors withdraw their claims for reasons unknown to the bank. This type of withdrawal is referred to as the *noisy withdrawal*. Second, depositors will also withdraw at date 1 if they calculate that it is worse off for them to hold the deposit contract to date 2 than to withdraw immediately. This type of withdrawal is referred to as the *rational withdrawal*. The difference between these two types of withdrawal is that while the bank can prevent the rational withdrawal by setting a high enough deposit rate r_2, the noisy withdrawal is beyond the its control and engenders a rigid liquidity demand. If the bank fails to meet all the withdrawal demands, a liquidity crisis occurs. We assume that the liquidity crisis is very costly to the bank. Hence, the bank makes sure that it meets all the withdrawal demands at date 1.

To meet the withdrawal demands, the bank a stock of G units of corn. In addition, it can also exchange loans for corn from investors. Following Bigio (2015), we allow two interpretations for this exchange: direct sale or collateralised borrowing. Let δ be the quantity of liquidity that it obtains in exchange of one unit of loans, or equivalently, δ is the discount factor that investors apply in evaluating the loans. We assume that at date 1, no one but the bank observes the quality of the loans represented by the realisation of $\widetilde{q}$. If $\widetilde{q} = q_L$ that is, if the bank finds its loans are lemons, it might want to dump them to investors. Hence, the liquidity borrowing of the bank is beset by the typical lemons problem. Following the literature on this problem (e.g. Bolton et al 2011 and Malherbe 2014), we assume that the investors do not observe the total quantity X of liquidity that the bank is obtaining, because this quantity signals the dumping of lemons. Specifically, we assume that if the bank wants to obtain X units of corn, then it randomly contact X out of N investors, to obtain one unit of liquidity from each. A given investor is thus contacted with probability X/N. An investor observes whether he has been contacted by the bank or not, but has no idea of how many other investors the bank is contacting or will contact. Therefore, if investors believe that the bank borrows X_H units of corn if it observes $\widetilde{q} = q_H$ and X_L if $\widetilde{q} = q_L$, then conditional on being contacted, they believe that the loans are lemons with the following probability:

$$\frac{p\frac{X_L}{N}}{p\frac{X_L}{N} + (1-p)\,\frac{X_H}{N}} = \frac{pX_L}{pX_L + (1-p)\,X_H}.$$

Then the investors use the following discount factor to evaluate the loans:

$$\delta = \frac{pX_L}{pX_L + (1-p)\,X_H}\delta_L + \frac{(1-p)\,X_H}{pX_L + (1-p)\,X_H}\delta_H. \tag{8}$$

A smaller discount factor δ represents a high cost for the bank to obtain external liquidity.

If the bank has lent out D units of liability at date 0, at date 1, then the size of noisy withdrawals is ωD. Given its liquidity stock G, to avoid the liquidity crisis, the quantity X of external liquidity to be obtained must satisfy $X \geq \underline{X}$, where

$$\underline{X} := \max\left(\omega D - G, 0\right). \tag{9}$$

Thus $\underline{X}$ represents the genuine liquidity need that the bank has to use external liquidity to satisfy. Given that the bank has DR units of loans, the maximum quantity of liquidity that it can obtain is

$$\overline{X} := DR\delta. \tag{10}$$

This $\overline{X}$ also represents the maximum scale of lemon dumping. The bank will not put itself in a situation where it will definitely fail to meet the liquidity demand, that is, the following condition holds in equilibrium:

$$\overline{X} \geq \underline{X}. \tag{11}$$

Following Heider et al (2015) and Parlour and Plantin (2008), we assume that the bank is not a Stackelberg leader to investors; that is, it cannot use its date-0 actions (D, R, r_2) to influence investors' decision on the discount factor δ at date 1. What matters for this decision is $\{\underline{X}, \overline{X}\}$ because, as we will show, $\{X_L, X_H\} \subset \{\underline{X}, \overline{X}\}$. Therefore, we assume that at date 1, investors observe neither $\underline{X}$ nor $\overline{X}$. The bank's depositors – i.e. workers who accept contract $(1, r_2)$ as the means of wage payment – certainly observe the deposit rate r_2 at date 0. We assume that they also observe (D, R) at date 1 when they decide on the rational withdrawal.[6]

The time line of the model is as follows.

Date 0.

1. The bank, endowed with G units of corn, decides the lending rate R and deposit rate r_2.

2. Entrepreneurs each borrow D units of the bank's liability contract $(1, r_2)$.

3. Entrepreneurs use the bank liability to hire D workers each and start production. Workers un-hired work in autarky.

Date 1.

1. The bank, but no one else, observes the realisation of $\widetilde{q}$. Depositors observe (D, R, r_2).

2. A fraction ω of depositors makes the noisy withdrawal; other depositors may opt for the rational withdrawal.

[6] The difference in date-1 information between depositors and investors is meant to represent the following facts. In reality, when an investor is contacted by a bank to buy an asset (or accept it as the collateral for lending liquidity), he will examine the asset and find a bundle of attributes that are informative of its quality; let us refer to these attributes as the *category*. In the presence of the lemons problem, he will calculate what is the chance that the bank has a genuine liquidity need and happens to pick this category of assets for sale; and what is the chance that the bank knows these assets are lemons and wants to dump them. The former chance is proportional to the quantity $\underline{X}$ of the liquidity need that the bank decides to meet by selling the category of assets, the latter to their total market value $\overline{X}$. The investor observing $\underline{X}$ or $\overline{X}$ amounts to him observing the detailed categorisation of the bank's balance sheet, which, in practice, is unlikely. By contrast, what concerns the rational withdrawal by depositors is the aggregate-level information, which is more readily available.

3. The bank obtains X units of the external liquidity from investors by selling the loans (or with collateralized borrowing). Investors observe neither X, nor $\left\{\underline{X}, \overline{X}\right\}$. They discount the loans with the discount factor δ.

Date 2.

1. The entrepreneurs produce corn. They repay $\min\left(\widetilde{A}D^{\alpha}, DR\right)$ units of corn to the bank to settle the loans. The bank redeems each unit of the outstanding deposits with a payment of r_2 whenever it can. If it cannot, it defaults and its asset is distributed pro rata to the depositors.

2. The economic agents derive utility from the corn that they have obtained.

The bank has monopolistic power over entrepreneurs and will earn a positive value ex ante. Hence, it will not choose such a course of action that it defaults in both states. Bank default could happen only in the bad state.

Definition 1 *Equilibrium of the model is a profile of* $\{D; R, r_2, X_L, X_H; \delta\}$ *that satisfies the following conditions.*

1. *Given the gross lending rate R, entrepreneurs' demand for deposits is $D = D(R)$ given by (5).*

2. *Given the discount factor δ used by investors and (D, R, r_2) chosen at date 0, the bank borrows X_H units of corn if it observes $\widetilde{q} = q_H$ and X_L units if it observes $\widetilde{q} = q_L$ at date 1.*

3. *Depositors make rational withdrawal if and only if the value of holding contract $(1, r_2)$ to date 2 is smaller than 1.*

4. *Given the discount factor δ, the demand function $D(R)$ and depositors' decision on the rational withdrawal, the bank makes the optimal decision on (R, r_2) at date 0.*

5. *The discount factor δ is determined by (X_L, X_H) via (8) whenever $(X_L, X_H) \neq (0, 0)$.*

We will examine the impact of the bank's liquidity stock G for its lending rate R in equilibrium. This impact, as will be shown, is channelled by the cost of the external liquidity, which the bank might need to meet its liquidity demand. As shown by Morrison and Wang (2019), the rational withdrawal can be costlessly prevented with a proper deposit rate r_2 alone and needs no liquidity to meet. Therefore, if there is no noisy withdrawal, the liquidity stock bears no impact on the bank's lending rate, as we show in the following benchmark case.

3 The Benchmark: No Noisy Withdrawal

Assume in this section that $\omega = 0$ so that the bank faces no noisy withdrawal. As a result, by (9), the genuine liquidity need of the bank $\underline{X} = 0$ for any D. Later in the next section we will show that in this scenario, the bank does not to borrow: $X_H = X_L = 0$; for the time being, we assume no borrowing by the bank at date 1 and focus on showing how the bank uses deposit rate r_2 to prevent the rational withdrawal costlessly.

At date 1, if the holder of one unit of deposit withdraws his claim, he obtains one unit of corn. If he holds the deposit to date 2, he will obtain r_2 in the good state. In the bad state, given no liquidity borrowing by the bank, the asset value of the bank is $DR \times \underline{A}/\left(\overline{A}\alpha\right) + G$ by Lemma 1. It has D units of outstanding deposit if the rational withdrawal is prevented. Hence the payment to each unit of deposit in the bad state, denoted by $\underline{r_2}$, is equal to the following.

$$\underline{r_2} = \min\left(r_2, \frac{DR \times \underline{A}/\left(\overline{A}\alpha\right) + G}{D}\right).$$

At date 1, depositors believe that the good state happens with probability q_e. Hence, the value of holding a unit of deposit to date 2, denoted by u_h, is as follows.

$$u_h\left(r_2\right) = q_e r_2 + \left(1 - q_e\right)\underline{r_2}.$$

The bank prevents the rational withdrawal at date 1 by offering a deposit rate r_2 that satisfies the following liquidity constraint:

$$u_h\left(r_2\right) \geq 1. \tag{12}$$

At date 0, if the rational withdrawal is prevented, the unit value of deposit is $\omega \times 1 + (1-\omega) \times u_h\left(r_2\right)$, which is hence the cost to the bank of issuing one unit of deposit. If the bank charges lending rate R, it lends out $D = D\left(R\right)$ units of deposits to entrepreneurs, who accordingly enter $D\left(R\right)R$ units of loans, each unit of which is of market value δ_e at date 0. The bank's problem at date 0 is hence as follows.

$$\max_{R,r_2} D\left(R\right)\left(R\delta_e - \left[\omega + (1-\omega)\, u_h\left(r_2\right)\right]\right), s.t. (12). \tag{13}$$

At the optimum, the liquidity constraint (12) binds: The optimal deposit rate r_2 satisfies $u_h\left(r_2\right) = 1$. As a result, at date 0 one unit of deposit is worth 1 and depositors break even. Hence the bank incurs no extra cost in using deposit rate r_2 to prevent the rational withdrawal. The optimal lending rate R^{NW} (here "NW" stands for "No Withdrawal") is:

$$R^{NW} = \frac{1}{\alpha} \times \frac{1}{\delta_e}. \tag{14}$$

To understand this equation, let us define the *marginal lending cost* as the value of the lending rate R at which the marginal profit of lending is 0. By (13), the marginal lending profit is equal to $R\delta_e - 1$. Hence, the marginal lending cost in the benchmark case

$$c^{NW} = \frac{1}{\delta_e}. \tag{15}$$

The optimal lending rate is thus equal to the marginal cost of lending multiplied by the mark up factor $1/\alpha$, which presents itself because the bank has monopolistic power over the borrowers. Observe that the bank's lending rate R^{NW} in the benchmark case is independent of its liquidity stock G. Hence,

Proposition 1 *If $\omega = 0$ and there is no noisy withdrawal, then the equilibrium lending rate $R^{NW} = 1/(\alpha\delta_e)$ and is unaffected by the bank's liquidity stock G.*

We have found that if the noisy withdrawal probability $\omega = 0$, then the bank's liquidity stock G bears no impact on the lending rate. In what follows, we assume the noisy withdraw is substantial:

$$\omega \geq 1 - \frac{\underline{A}}{\overline{A}\alpha}; \tag{16}$$

and show that in this case the liquidity stock has a non-monotonic and discontinuous relationship with the lending rate. To a large extent, this relationship is driven by the *double identity* of liquid assets, namely the fact that they are both liquid assets and safe assets.

To characterize the equilibrium, we first examine the decision problems of different parties separately, and then the meeting of these decisions in equilibrium. For the former task, we use backward induction. Given that at date 2 no decision is made, only contracts settled, we start with date 1 decisions and then move back to date-0 decisions.

4 Date-1 Decisions

At date 1, given (D, R, r_2) that has been determined at date 0 and the investors' discount factor δ, the bank decides the quantity X_H or X_L of the external liquidity to borrow from investors conditional on the realisation of $\widetilde{q}$. Depositors, based on their observation of (D, R, r_2) and their belief of δ, decide whether to make the rational withdrawal. Finally, investors, based on their rational expectation of (D, R, r_2), decide the discount factor δ. We analyse these three decisions in order.

4.1 The liquidity borrowing by the bank

If the bank decides to borrow $X \in [\underline{X}, \overline{X}]$ units of liquidity (i.e. corn), it surrenders X/δ units of loans as the collateral, hence retaining $DR - X/\delta$ units of loans on its balance sheet. Then it has $X + G$ units of corn, out of which ωD is used to meet the demand of the noisy withdrawal; the rational withdrawal, as was shown in the benchmark case above, is prevented by the bank using a proper depositor rate r_2. At date 2, if the state is good, the loans perform, each unit of which returns 1 unit of corn. Hence, the asset value is $DR - X/\delta + X + G - \omega D$. The bank pays r_2 to redeem each of $(1-\omega) D$ units of the outstanding liability in full. Hence, the bank's value is

$$V_G(X) = DR - \frac{X}{\delta} + X + G - \omega D - (1-\omega) Dr_2, \tag{17}$$

and a unit of deposit is repaid with r_2; it is convenient to find the unit repayment to deposit here so that later we can readily write the liquidity constraint about the deposit rate r_2 for stopping the rational withdrawal. In the bad state, the loans do not perform, each unit of which returns $\underline{A}/\left(\overline{A}\alpha\right) < 1$. If the bank pays deposits in full, then its value is

$$\widetilde{V}_B(X) = \left(DR - \frac{X}{\delta}\right)\frac{\underline{A}}{\overline{A}\alpha} + X + G - \omega D - (1-\omega) Dr_2. \tag{18}$$

If $\widetilde{V}_B(X) < 0$, the bank defaults and its value is zero. Thus, the bank's value is $\max\left(\widetilde{V}_B(X), 0\right)$ and the repayment to a unit deposit is a function of X as follows.

$$\begin{aligned}\underline{r_2}(X) &= \min\left(r_2, \frac{\left(DR - \frac{X}{\delta}\right)\frac{\underline{A}}{\overline{A}\alpha} + X + G - \omega D}{(1-\omega) D}\right) & (19)\\ &= r_2 + \min\left(0, \frac{\widetilde{V}_B(X)}{(1-\omega) D}\right). & (20)\end{aligned}$$

If the bank defaults and $\widetilde{V}_B(X) < 0$, then by equation (20), $-\widetilde{V}_B(X)/[(1-\omega) D]$ is the shortfall in the deposit repayment r_2, or the loss borne by a unit deposit in the bad state. At date 1, conditional on the realisation of $\widetilde{q}$, the bank's decision on the quantity of the external liquidity X to obtain is as follows.

$$\max_{X \in [\underline{X}, \overline{X}]}\left[V(X) := \widetilde{q}V_G(X) + (1-\widetilde{q})\max\left(\widetilde{V}_B(X), 0\right)\right]. \tag{21}$$

It follows that

$$V'(X) = \left\{\begin{array}{c} -\delta(\widetilde{q})/\delta + 1 \text{ if } \widetilde{V}_B(X) > 0 \\ \widetilde{q}(-1/\delta + 1) < 0 \text{ if } \widetilde{V}_B(X) < 0 \end{array}\right\}. \tag{22}$$

Behind equation (22) stand two effects. First is the *asset-value effect*. The true value of a unit of loan is $\delta(\widetilde{q})$, but investors believe it is worth δ. Given $\delta \in [\delta_L, \delta_H]$, investors undervalue bank loans if $\widetilde{q} = q_H$ and over-value them while if $\widetilde{q} = q_L$. Thus, giving up $1/\delta$ units of loans for one unit of liquidity changes the bank's asset value by $-\delta(\widetilde{q})/\delta + 1$. If the bank never defaults (i.e. $\widetilde{V}_B(X) > 0$), then this change is all accrued to the bank (namely, the equity): $V'(X) = -\delta(\widetilde{q})/\delta + 1$. However, if $\widetilde{V}_B(X) < 0$ and the bank defaults in the bad state, the second effect enters the stage. Note that loans are an risky asset and the liquid asset – i.e. corn – is risk free. Therefore, an exchange of loans for liquidity swaps the risky assets with the risk free one and amounts to *inverse risk-shifting*. That, following the seminal work of Jensen and Meckling (1976), increases the value of the debt and decreases the value of the equity when the debt is risky: $V'(X) < 0$ if $\widetilde{V}_B(X) < 0$ according to (22). The equity's loss due to inverse risk-shifting is the flip side of its gain due to risk-shifting. Hence, if the loss $-\widetilde{V}_B(X) / [(1-\omega) D]$ borne by deposits is larger, as the bank's gain from risk-shifting becomes greater, so does its loss from inverse risk-shifting.

The inverse risk-shifting effect pushes bank to choose a smaller X, always. So does the asset value effect if $\widetilde{q} = q_H$. Hence the following lemma.

Lemma 2 $X_H = \underline{X}$, *that is, if the loan quality is high, the bank borrows exactly what is needed to meet its liquidity demand.*

Proof. If $\widetilde{q} = q_H$, $V'(X) < 0$ by (22). Hence X_H hits the lower bound, i.e. $X_H = \underline{X}$. ■

The following corollary of this lemma is used to in the benchmark case.

Corollary 2 *If* $\underline{X} = 0$ *for any* D*, then* $X_H = X_L = 0$.

Proof. If $\underline{X} = 0$ for any D, then investors know that $X_H = 0$. Then by (8), $\delta = \delta_L$. It follows from (22) that $V'(X) \leq 0$ if $\widetilde{q} = q_L$. Hence, $X_L = 0$. ■

The proof of the corollary also shows that if $\delta = \delta_L$, then

$$X_H = X_L = \underline{X}.$$

We now consider the case in which $\delta > \delta_L$. In this case, if $\widetilde{q} = q_L$ investors strictly over-price bank loans and the asset-value effect drives the bank to dump all the lemons – namely low-quality loans – to investors and choose $X_L = \overline{X}$. However, the inverse risk-shifting effect still drives the bank to borrow the minimum: $X_L = \underline{X}$. The choice of X depends on the balance between these two effects. The latter effect is stronger, we just observed, if the bad-state loss borne by deposits is greater. The liquidity stock G, as the safe asset, absorbs loss for deposits. The greater the safe asset G, the smaller the loss borne by deposits and the weaker the inverse risk-shifting effect. A greater G, therefore, tilts the balance to the lemon dumping end: $X_L = \overline{X}$. This intuition is confirmed by Lemma 3 below.

Lemma 3 *If* $\delta > \delta_L$*, then*

$$X_L = \left\{ \begin{array}{c} \overline{X} \text{ if } G > \gamma D \\ \{\underline{X}, \overline{X}\} \text{ if } G = \gamma D \\ \underline{X} \text{ if } G < \gamma D \end{array} \right\}, \tag{23}$$

where

$$\gamma = \gamma(R, r_2, \delta) := \omega - \left(R - \frac{(1-q_L)(1-\omega)}{\delta - q_L} r_2 \right) \delta. \tag{24}$$

Moreover, $\partial\gamma/\partial R < 0$, $\partial\gamma/\partial\delta < 0$ *and* $\gamma < \omega$. *Lastly, if* $X_L = \overline{X}$, *then* $\widetilde{V}_B(X_L) > 0$; *and if* $X_L = \underline{X}$ *then* $\widetilde{V}_B(X_L) < 0$.

Proof. See Appendix A. ■

By this lemma, if $\delta > \delta_L$, the bank's liquidity acquiring in the contingency of $\widetilde{q} = q_L$ falls into two regimes. In *Regime A*, the bank chooses $X_L = \underline{X}$ abstaining from lemon dumping. This regime rules if the $D > G/\gamma$, that is, the bank's lending scale D is large enough relative to its safe asset G. Under this condition, the loss borne by the deposits is large enough so that the inverse risk-shifting effect is strong enough to overcome the lemons problem. Hence, the bank borrows the minimum necessary quantity of the external liquidity when $\widetilde{q} = q_L$, as it does when $\widetilde{q} = q_H$. In *Regime B*, the bank dumps lemons and borrows $\overline{X}$. This regime rules if the lending scale $D < G/\gamma$ and hence the inverse risk-shifting effect is too weak to overcome the lemons problem. Observe that in determining the regime choice by the bank, the corn stock G works in the identity of the safe asset.

Variable γ represents the quantity of the corn stock per unit of lending at which the loss due to the inverse risk-shifting is exactly balanced with the gain from dumping lemons. Note that $\partial\gamma/\partial R < 0$. Intuitively, if the gross lending rate R rises, each unit of lending generates more units of loans and thus entails a greater benefit from dumping them when they are lemons. To balance that, the inverse risk shifting effect needs to be stronger, namely, the bad-state loss borne by deposits needs to be higher. Therefore, the safe asset γ that cushions each unit of deposits need be smaller.

4.2 The rational withdrawal by depositors

We have found that at date 2, a unit deposit pays off r_2 in the good state and $\underline{r_2}$ in the bad state which depends on the bank's borrowing scale X via (19). Now we consider depositors' decision on the rational withdrawal. At date 1, depositors observe (D, R, r_2) and form a rational belief of δ, whereby they deduce (X_H, X_L). With probability p, the quality shock $\widetilde{q} = q_L$ realises and the bank borrows X_L units of the external liquidity. With probability $1-p$, the scale is $X_H = \underline{X}$. Holding one unit of deposit to date 2 thus delivers the following payoff:

$$\begin{aligned} u_h(r_2) &= p\left[q_L r_2 + (1-q_L)\underline{r_2}(X_L)\right] + (1-p)\left[q_H r_2 + (1-q_H)\underline{r_2}(\underline{X})\right] \\ &= q_e r_2 + p(1-q_L)\underline{r_2}(X_L) + (1-p)(1-q_H)\underline{r_2}(\underline{X}). \end{aligned} \tag{25}$$

Withdrawal at date 1 delivers 1 unit of corn. The rational withdrawal is prevented if and only if the deposit rate r_2 satisfies that following liquidity constraint:

$$u_h(r_2) \geq 1. \tag{26}$$

Obviously, $u_h(r_2)$ increases with r_2. Hence, the above liquidity constraint is satisfied if and only if $r_2 \geq r_2^*$, where threshold r_2^* makes the constraint bind:

$$u_h(r_2^*) = 1. \tag{27}$$

As the value $u_h(r_2)$ of holding deposits to date 2 depends on the borrowing scale X, so does the threshold r_2^*. From the preceding subsection, we know $X_L = \underline{X}$ in Regime A and $X_L = \overline{X}$ in Regime B. Hence, the thresholds r^* in the two regimes are different. Moreover, the former is the case if $\delta = \delta_L$ or $G \leq \gamma(R, r_2, \delta) D$, the latter if $\delta > \delta_L$ and $G \geq \gamma(R, r_2, \delta) D$. Hence the following lemma.

Lemma 4 *There exist two threshold functions $r^A(R, \delta, G)$ and $r^B(R, \delta, G)$ such that depositors make no rational withdrawal if $r_2 \geq r^A$ when they expect $X_L = \underline{X}$ and if $r_2 \geq r^B$ when they expect $X_L = \overline{X}$. That is,*

$$r_2^* = \left\{ \begin{array}{c} r^A(R, \delta, G) \ \text{ if } \delta = \delta_L \text{ or } G \leq \gamma(R, r_2, \delta) D(R) \\ r^B(R, \delta, G) \ \text{ if } \delta > \delta_L \text{ and } G \geq \gamma(R, r_2, \delta) D(R) \end{array} \right\}. \tag{28}$$

Both $r^A(R, \delta, G)$ and $r^B(R, \delta, G)$ decrease with (R, δ, G). Lastly $r^A(R, \delta, G) > r^B(R, \delta, G)$.

Proof. See Appendix A. ■

Both thresholds r^A and r^B decrease with (R, δ, G) because the unit deposit payoff $\underline{r}$ in the contingency of bank default increases with (R, δ, G), for the following intuitive reasons. If the lending rate R is higher, the bad-state return $\underline{A}/(\overline{A}\alpha) \times R$ of lending per unit of deposits is higher and thus so is the unit deposit repayment $\underline{r}$. Similarly, the larger the liquidity stock G, the greater the loss absorbed by the safe asset in the bad state and the higher the unit deposit repayment $\underline{r}$. Lastly, the bigger the discount factor δ, the higher the value of the bank's asset and the higher is $\underline{r}$.

The bank exchanges less loans for liquidity in Regime A than in Regime B. Due to the inverse risk-shifting effect, given r_2, the value of holding deposits in Regime A is lower than in Regime B. As a result, to stop the rational withdrawal, the deposit rate r_2 needs to be higher in Regime A than in Regime B. Hence, $r^A > r^B$.

In the knife-edge scenario where $\delta > \delta_L$ and $G = \gamma(R, r_2, \delta) D$, by Lemma 3, the bank could play a mixed strategy with X_L, in which case the threshold r_2^* is somewhere between r^A and r^B. However, in this paper, we confine our attention to the case where *in equilibrium* the bank plays no mixed strategy.

4.3 The discount factor used by investors

At date 1, based on their rational expectation of (D, R, r_2) chosen at date 0 and their observation of G, investors decide the discount factor δ that they use to evaluate the loans that the bank offers. For this purpose, they deduce (X_H, X_L) and then find δ using equation (8) whenever $(X_H, X_L) \neq (0, 0)$. Their finding is given by Proposition 3 below, to state which it is convenient to state the following lemma beforehand.

Lemma 5 *Equation (29) below has a unique solution for δ within interval (δ_L, δ_e), denoted by $\overline{\delta_B}$.*

$$\frac{\delta - \delta_L}{\delta_H - \delta} = \frac{1-p}{p}\left[1 - \frac{(1-q_L)(1-\omega)}{\delta - q_L}\frac{r_2}{R}\right]. \tag{29}$$

Moreover, $\overline{\delta_B}'\left(\frac{r_2}{R}\right) < 0$.

Proof. See Appendix A. ■

In Proposition 3 below, we suppress arguments (R, r_2) of function $\gamma(R, r_2, \delta)$ defined by (24). The proposition uses the fact that $\partial\gamma/\partial\delta < 0$ and $\gamma < \omega$ as given in Lemma 3.

Proposition 3 *Given investors' rational expectation of (D, R, r_2), the discount factor δ that they use to evaluate loans at date 1 is a continuous function of the liquidity stock G, as follows.*

$$\delta = \left\{ \begin{array}{c} \delta_e \text{ if } G \leq \gamma(\delta_e) D \\ \delta_M \text{ if } G \in \left[\gamma(\delta_e) D, \gamma\left(\overline{\delta_B}\right) D\right] \\ \delta_B \text{ if } G \in \left[\gamma\left(\overline{\delta_B}\right) D, \omega D\right] \\ \delta_L \text{ if } G \geq \omega D \end{array} \right\}, \tag{30}$$

where δ_M is the unique root of

$$\gamma(\delta) D = G;$$

and δ_B is a unique root of

$$\frac{\delta - \delta_L}{\delta_H - \delta}\delta = \frac{1-p}{p}\frac{\omega D - G}{DR}; \tag{31}$$

and at $G = \gamma(\delta) D$, equation (31) is reduced to (29) and hence $\delta_B = \overline{\delta_B}$. Moreover δ decreases G.

Proof. See Appendix A. ■

The first branch of equation (30) represents that case where the bank is in Regime A and contacts an investor with an equal probability in the contingency of $\widetilde{q} = q_L$ as in that of $\widetilde{q} = q_H$ (namely $X_L = X_H$). Therefore, $\delta = \delta_e$ as if the lemons' problem were absent. By Lemma 3, Regime A rules if $G < \gamma(\delta) D$, which, with $\delta = \delta_e$, is equivalent to $G < \gamma(\delta_e) D$. On the other hand, the third branch of equation (30) represents that case where the bank is in Regime B and thus $X_L = \overline{X}$. Substituting this and $X_H = \underline{X} = \omega D - G$ into equation (8) finds equation (31). With the endogeneity of δ and Lemma 3, this is the case if $G > \gamma\left(\overline{\delta_B}\right) D$. Between these two cases, there is a gap of G, namely, $\left[\gamma\left(\delta_e\right) D, \gamma\left(\overline{\delta_B}\right) D\right]$.[7] When G ascends through this gap, the probability of the bank playing $X_L = \overline{X}$ rises from 0 to 1, so that the case of Regime A smoothly transitions into that of Regime B. In this transition case, represented by the second branch of equation (30), the bank plays a mixed strategy with X_L, which demands $G = \gamma\left(\delta\right) D$, that is, $\delta = \delta_M$. Lastly, the fourth branch of equation (30) represents the case where $\underline{X} = \max\left(\omega D - G, 0\right) = 0$. In this case investors believe that the bank needs no external liquidity to meet its liquidity demand. They interpret any of its offers to exchange loans for liquidity as an attempt to dumping lemons and set $\delta = \delta_L$. Observe that while *given* $\left(D, R, r_2\right)$, there is always an interval of G within which the bank plays a mixed strategy with X_L. However, in equilibrium, where the dependence of $\left(D, R, r_2\right)$ on G is taken into account, in no interval of G is a mixed strategy played in the case that interests us.

Overall, by Proposition 3 δ decreases with G, as illustrated in the following figure.

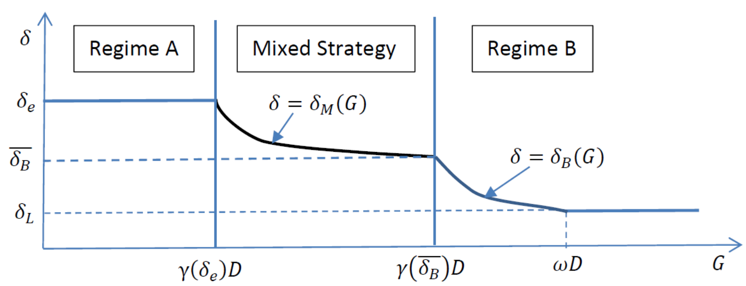

Figure 1: Given $\left(D, R, r_2\right)$, the discount factor δ continuously decreases with the liquidity stock G. If $G \leq \gamma\left(\delta_e\right) D$, the bank is in Regime A where the inverse risk-shifting effect overcomes the lemons problem. If $G \geq \gamma\left(\overline{\delta_B}\right) D$, the bank is in Regime B where it dumps lemons. In between is the case where the bank plays a mixed strategy with X_L.

In the decreasing relationship of δ with G, the corn stock G is in both identities of the safe asset and the liquid asset. First, we have seen above that it is in identity of the safe asset that G determines the borrowing regime of the bank. Second, in the decreasing relationship of δ with G in Regime B, the corn stock works as the liquid asset. In this regime, the bank dumps lemons when $\widetilde{q} = q_L$, but when $\widetilde{q} = q_H$, it borrows the quantity of liquidity just sufficient to meet its needs. The larger is its own liquid stock G, the less the external liquidity needed. Consequently, when an investor is contacted by the bank, the less likely is this contact made because of the liquidity needs, the more likely is it out of the lemon-dumping purpose and the lower the discount factor δ_B.

[7]Observe that by Lemma 5 $\delta_e > \delta_B$ and hence $\gamma\left(\delta_e\right) < \gamma\left(\overline{\delta_B}\right) D$.

5 Date-0 Decisions

At date 0, the demand of bank deposits by entrepreneurs is $D = D(R)$ given in Lemma 1. A unit-deposit bearer expects to withdraw one unit of corn (for reasons unknown to the bank) with probability ω and holds the deposit on to date 2 with probability $1-\omega$. Ex ante a unit of deposit is worth $u_0 = \omega \times 1 + (1-\omega) u_h$. $u_0 \geq 1$ because of the date-1 liquidity constraint $u_h \geq 1$. Therefore, workers do not hesitate accepting a unit of deposit as the wage payment at date 0. Only the bank's decision on (R, r_2) awaits to be examined. In making these decisions, it takes as given the discount factor δ that investors will use at date 1. By Proposition 3, $\delta \in [\delta_L, \delta_e]$.

The bank's optimal choice of the deposit rate r_2 can be easily characterized. To stop depositors from making the rational withdraw, $r_2 \geq r_2^*$, where r_2^* is a function of (R, δ, G) given by (28). At the optimum, $r_2 = r_2^*(R, \delta, G)$ and the liquidity constraint binds, namely $u_h(r_2) = 1$. As a result, in equilibrium the value of a unit of deposits $u_0 = 1$. If the bank lends out D units of deposits at date 0, then its date-0 value is the expected asset value V minus D.

The asset value V depends on the lending rate R as follows. At the lending rat R, the lending scale is $D = D(R)$. At date 1, conditional on the quality shock $\widetilde{q}$, the bank will exchange $X(\widetilde{q})/\delta$ units of loans for $X(\widetilde{q})$ units of liquidity. Each unit of the remained $DR - X(\widetilde{q})/\delta$ units of loans returns 1 in the good state and $\underline{A}/\left(\overline{A}\alpha\right)$ in the bad state at date 2. Hence, conditional on the quality shock $\widetilde{q}$, the expected asset value is

$$\begin{aligned}
&\widetilde{q}\left(DR - \frac{X(\widetilde{q})}{\delta} + X(\widetilde{q}) + G\right) + (1-\widetilde{q})\left(\left(DR - \frac{X(\widetilde{q})}{\delta}\right)\frac{\underline{A}}{\overline{A}\alpha} + X(\widetilde{q}) + G\right) \\
= &\left(DR - \frac{X(\widetilde{q})}{\delta}\right)\delta(\widetilde{q}) + X(\widetilde{q}) + G.
\end{aligned}$$

At date 0, the expected asset value is

$$V = p\left[\left(DR - \frac{X_L}{\delta}\right)\delta_L + X_L + G\right] + (1-p)\left[\left(DR - \frac{X_H}{\delta}\right)\delta_H + X_H + G\right]. \tag{32}$$

While $X_H = \underline{X} = \max(\omega D - G, 0)$ always by Lemma 2, the value X_L depends on the borrowing regime at date 1. In Regime A, $X_L = \underline{X}$ and the bank's asset value is

$$V^A = \left(DR - \frac{\underline{X}}{\delta}\right)\delta_e + \underline{X} + G, \tag{33}$$

and in Regime B, $X_L = \overline{X} = DR\delta$ and the asset value is

$$V^B = pDR\delta + (1-p)\left[\left(DR - \frac{\underline{X}}{\delta}\right)\delta_H + \underline{X}\right] + G. \tag{34}$$

As V^A and V^B differs only in the liquidity acquiring scale X_L in the contingency of $\widetilde{q} = q_L$. If $\delta = \delta_L$, the lemon loans are fair priced and this difference produces no effect to the bank's asset value: $V^A = V^B$.

Hence, in what follows, we focus on the case where $\delta > \delta_L$. In this case, the bank chooses $X_L = \underline{X}$ and is in Regime A if $\gamma D \geq G$; and chose $X_L = \overline{X}$ and is in Regime B if $\gamma D \leq G$. Considering that the $D = D(R)$ is negatively related to the lending rate R, we expect that Regime A is feasible if R is low enough, Regime B the opposite. This intuition is confirmed by the following lemma, where

$$\overline{R}(G) := \overline{A}\alpha \left(\frac{\omega}{G}\right)^{1-\alpha} \tag{35}$$

denote the inverse function of $G = \omega D(R)$, that is, $\underline{X} = \omega D(R) - G > 0$ if and only if its lending rate $R < \overline{R}(G)$.

Lemma 6 *Given G and $\delta > \delta_L$, there exists two thresholds $R^A(G, \delta)$ and $R^B(G, \delta)$ such that the bank can be in Regime A if $R \leq R^A(G, \delta)$ and in Regime B if $R \geq R^B(G, \delta)$. Moreover, $\overline{R}(G) > R^A(G, \delta) > R^B(G, \delta)$, where*

Proof. See Appendix A. ■

By this lemma, if $\delta > \delta_L$, for any $R \in [R^B, R^A]$, both regimes are feasible. This result is driven by an interaction between the bank's liability level and its borrowing regime. By Lemma 4, $r^A > r^B$. Given $R \in [R^B, R^A]$, if the bank chooses $r_2 = r^A(R, \delta, G)$ and offers the higher deposit rate at date 0, then other things equal, the bad-state shortfall in the deposit repayment is large. Consequently, the risk-shifting effect is strong enough to overcome the lemons problem, and the bank will indeed be in Regime A at date 1 and need the higher deposit rate r^A to stop the rational withdrawal. If the bank chooses $r_2 = r^B(R, \delta, G)$ and offers the lower deposit rate at date 0, then the risk-shifting effect is too weak to overcome the lemons problem. Consequently the bank will indeed be in Regime B at date 1 and $r_2 = r^B$ suffices to stop the rational withdrawal. Between these two regimes, the bank picks the one that gives a higher asset value, because the unit cost of liability is always 1 at date 0. That is Regime B, because the bank dumps lemons in the regime and dumping lemons increases its asset value if $\delta > \delta_L$.

Lemma 7 *If $\delta > \delta_L$, then $V^A < V^B$ for $R \in [R^B, R^A]$.*

Proof. See Appendix A. ■

By Lemma 7, if $\delta > \delta_L$, the bank's asset value $V = V^B$ whenever Regime B is feasible, that is, $R \geq R^B(G, \delta)$. Hence, if $\delta > \delta_L$,

$$V = \left\{ \begin{array}{l} V^A \text{ if } R < R^B(G, \delta) \\ V^B \text{ if } R \geq R^B(G, \delta) \end{array} \right\}. \tag{36}$$

We have seen that $V^A = V^B$ if $\delta = \delta_L$. It follows that equation (36) holds true for all $\delta \in [\delta_L, \delta_e]$.

At date 0, the bank's value $\Pi = V - D$. With V^A given by (33) and V^B by (18), the bank's value Π depends on the lending rate R as follows:

$$\Pi(R;\delta,G) = \left\{ \begin{array}{c} D(R)\left[\delta_e R - \left(\omega \frac{\delta_e - \delta}{\delta} + 1\right)\right] + \frac{\delta_e}{\delta} G \text{ if } R < R^B(G,\delta) \\ \begin{array}{c} D(R)\left[(p\delta + (1-p)\delta_H)R - \left(\omega \frac{(1-p)(\delta_H - \delta)}{\delta} + 1\right)\right] \\ + \frac{\delta_e + p(\delta - \delta_L)}{\delta} G \end{array} \text{ if } R \in [R^B(G,\delta), \overline{R}(G)] \\ D(R)\left[(p\delta + (1-p)\delta_H)R - 1\right] + G \text{ if } R \geq \overline{R}(G) \end{array} \right\}. \quad (37)$$

In the benchmark case, we have defined the lending cost as the value of R at which the profit margin of lending is equal to 0. Then, the three branches of equation (37) represent three lending scenarios, characterized by different lending costs. First is *Scenario A*, where the lending rate is at the lower end – $R < R^B$ – and thus the lending scale is large relative to G. The large scale lands the bank in Regime A, where it borrows $\underline{X} > 0$ (because $R^B < \overline{R}$) units of liquidity independent of the realisation of $\widetilde{q}$. Each unit of loans is hence worth δ_e. Lending out one unit of liability creates R units of loans and needs the service of ω unit of the external liquidity, each unit of which changes the bank's asset value by $(\delta_e - \delta)/\delta$. In Scenario A, hence, the profit marginal of lending is $\delta_e R - (1 + \omega(\delta_e - \delta)/\delta)$ and the marginal lending cost of lending is

$$c_A = \frac{1 + \omega(\delta_e - \delta)/\delta}{\delta_e}. \quad (38)$$

Second is *Scenario B1*, where the lending rate $R \in [R^B, \overline{R}]$ and is in a middle range. As a result, on the one hand, the lending scale is small enough to get the bank in Regime B. On the other hand, it is big enough that still $\underline{X} > 0$ and the bank needs the external liquidity to meet its liquidity demand. At date 1, the bank dumps all the loans if they are lemons. Hence, a unit of loan is worth δ_H when $\widetilde{q} = q_H$ and δ when $\widetilde{q} = q_L$ at date 1 and worth $p\delta + (1-p)\delta_H$ at date 0. Moreover, the bank needs to exchange loans for the external liquidity in the contingency of $\widetilde{q} = q_H$. Each unit of liquidity exchanged incurs a loss of $(\delta_H - \delta)/\delta$ to the bank. The contingency of $\widetilde{q} = q_H$ occurs with probability $1 - p$. Altogether, in Scenario B1, the profit margin of lending is $(p\delta + (1-p)\delta_H)R - (1 + (1-p)\omega(\delta_H - \delta)/\delta)$ and the marginal cost of lending is

$$c_{B1} = \frac{1 + (1-p)\omega(\delta_H - \delta)/\delta}{p\delta + (1-p)\delta_H}. \quad (39)$$

Third is *Scenario B2*, where the lending rate $R > \overline{R}$ and consequently $\underline{X} = 0$ and the bank can self-satisfy its liquidity demand. As in Scenario B1, the bank still dumps lemons and a unit of loan is worth $p\delta + (1-p)\delta_H$. Different to Scenario B2, the bank avoids the loss from borrowing liquidity in the contingency of $\widetilde{q} = q_H$. Therefore, in Scenario B2, the marginal cost of lending is

$$c_{B2} = \frac{1}{p\delta + (1-p)\delta_H}. \quad (40)$$

These three marginal costs are different, as shown by the lemma below.

Lemma 8 *$c_A \geq c_{B1} > c_{B2}$ for any $\delta \in [\delta_L, \delta_e]$ and the equality holds if and only if $\delta = \delta_L$.*

Proof. See Appendix A. ■

It is obvious that $c_{B1} > c_{B2}$. The reason for $c_A \geq c_{B1}$ is the same as that drives $V^B \geq V^A$: At date 0, dumping more lemons helps the bank reduce the lending cost, unless $\delta = \delta_L$. By this lemma, the marginal cost change discontinuously across scenarios. As a result, the value function of the bank $\Pi(R)$, though continuous everywhere, has a kink at the two boundaries between these three scenarios: $R = R^B(G, \delta)$ and $R = \overline{R}(G)$.

The boundary $\overline{R}(G) = \overline{A}\alpha(\omega/G)^{1-\alpha}$ is simple enough. The boundary $R^B(G, \delta)$ is the lending rate at which the bank is indifferent between the two regimes at date 1 if it has chosen $r_2 = r^B(R, \delta, G)$, that is, $R^B(G, \delta)$ is implicitly defined by

$$\gamma\left(R, r^B(R, \delta, G), \delta\right) D(R) = G. \tag{41}$$

Lemma 9 *$R_G^{B\prime} < 0$ and $R_\delta^{B\prime} < 0$.*

Proof. See Appendix A. ■

Intuitively, at the boundary rate R^B, the lending scale is at the level where the gain from lemon dumping is exactly balanced with the loss from inverse risk-shifting. In the decreasing relationship of R^B with G, the corn stock G works in the identity of the safe asset. A larger safe asset G, given the lending scale, weakens the inverse risk-shifting effect, in order for which to balance the lemons problem, therefore, the lending scale needs be raised, and the lending rate reduced. Hence, $R_G^{B\prime} < 0$. A higher discount fact δ, similarly, increases the gain from lemon dumping, to balance which, the inverse risk-shifting effect needs get stronger. Therefore, the lending scale needs be raised, and the lending rate R^B reduced. Hence, $R_\delta^{B\prime} < 0$.

While function $R^B(G, \delta)$ cannot be explicitly spelt, its inverse function $G = \Gamma(R, \delta)$ can, which is also implicitly defined by equation (41). The key is that at $R = R^B(G, \delta)$, r^B is independent of G, as shown in the following lemma.

Lemma 10

$$r^B\left(R^B, \delta, G\right) = \frac{1}{1 - (1-p)(1-q_H)\frac{\delta - \delta_L}{\delta - q_L}}. \tag{42}$$

The inverse function of $R^B(\cdot, \delta)$ is

$$\Gamma(R, \delta) = \left[\omega - \left(R - \frac{(1-q_L)(1-\omega)}{(1-(1-p)(1-q_H))(\delta - \delta_L) + \delta_L - q_L}\right)\delta\right] D(R). \tag{43}$$

Moreover, $\Gamma(R, \delta) < \omega D(R)$ for any $R > 1$; and $\Gamma_R' < 0$ and $\Gamma_\delta' < 0$.

Proof. See Appendix A. ■

Given the discounter factor δ and the liquidity stock G, the bank's decision problem at date 0 is therefore:

$$\max_{R} \Pi\left(R; \delta, G\right). \tag{44}$$

Its solution is given in the following proposition where we use the fact that $\Gamma\left(\frac{1}{\alpha}c_A, \delta\right) \leq \Gamma\left(\frac{1}{\alpha}c_{B1}, \delta\right)$, which holds because $\Gamma'_R < 0$ and $c_A \geq c_{B1}$ by Lemma 8.

Proposition 4 *Given (δ, G), the optimal lending rate R^* that the bank charges at date 0 is as follows.*

$$R^* = \left\{ \begin{array}{c} \frac{1}{\alpha}c_A\left(\delta\right) \ \textit{if } G < \Gamma\left(\frac{1}{\alpha}c_A\left(\delta\right), \delta\right) \\ R^B\left(G, \delta\right) \ \textit{if } G \in \left[\Gamma\left(\frac{1}{\alpha}c_A\left(\delta\right), \delta\right), \Gamma\left(\frac{1}{\alpha}c_{B1}\left(\delta\right), \delta\right)\right] \\ \frac{1}{\alpha}c_{B1}\left(\delta\right) \ \textit{if } G \in \left[\Gamma\left(\frac{1}{\alpha}c_{B1}\left(\delta\right), \delta\right), \omega D\left(\frac{1}{\alpha}c_{B1}\left(\delta\right)\right)\right] \\ \overline{R}\left(G\right) \ \textit{if } G \in \left[\omega D\left(\frac{1}{\alpha}c_{B1}\left(\delta\right)\right), \omega D\left(\frac{1}{\alpha}c_{B2}\left(\delta\right)\right)\right] \\ \frac{1}{\alpha}c_{B2}\left(\delta\right) \ \textit{if } G \geq \omega D\left(\frac{1}{\alpha}c_{B2}\left(\delta\right)\right). \end{array} \right\}, \tag{45}$$

where c_A, c_{B1}, and c_{B2} are respectively given by (38), (39) and (40).

Proof. See Appendix A. ■

The five branches of equation (45) represents five phases of the optimal lending rate R^*. They are related to the three scenarios described above. With the corn stock G rising, due to its double identity, the inverse risk-shifting effect becomes weaker and the liquidity stock larger. Hence, bank lending naturally proceeds from Scenario A (where the inverse risk-shifting effect is strong enough to overcome the lemons problem), to Scenario B1 (where the lemons problem bites), and lastly to Scenario B2 (where the liquidity demand can be self-satisfied). These are respectively corresponding Phases 1, 3 and 5, where the optimal lending rate is equal to the marginal lending cost in the scenario, c_A, c_{B1} and c_{B2}, marked up with the factor $1/\alpha$. Phases 2 and 4 are the boundary cases, where R^* is equal to the respective boundary rate $R^B\left(G, \delta\right)$ and $\overline{R}\left(G\right)$. The boundary phase exists for an interval of G rather than a point of it because the lending cost changes abruptly across the scenarios. When G ascends through these two intervals, αR^* smoothly decreases, respectively, from c_A to c_{B1}, and from c_{B1} to c_{B2}.

For a $\delta > \delta_L$, Proposition 4 is illustrated as follows.

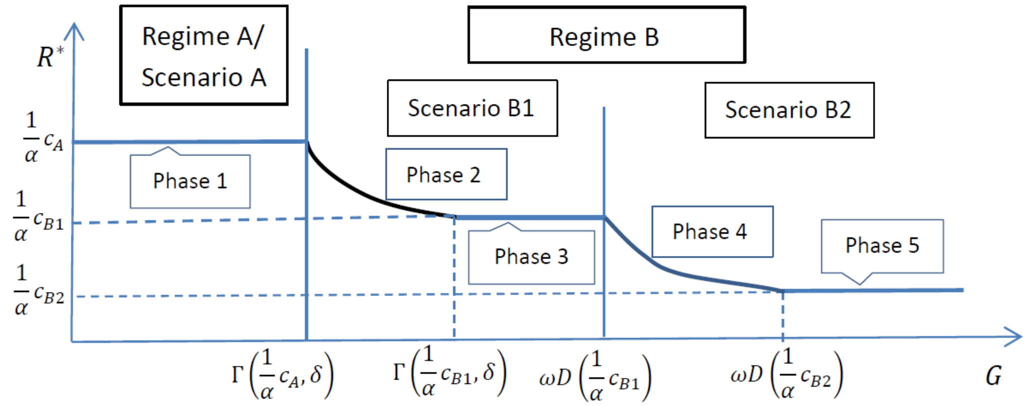

Figure 2: Given $\delta > \delta_L$, the optimal lending rate R^* continuously decreases with the liquidity stock G. In Phases 1, 3 and 5, the bank is inside Scenarios A, B1 and B2 and R^* is equal to the marginal lending cost marked up by $1/\alpha$. The transition between the scenarios happens smoothly in Phases 2 and 4, where $R^* = R^B(G, \delta)$ and $R^* = \overline{R}(G)$ respectively.

The monotonic and smooth relationship of R^* with G depicted in Figure 2 exists only because the discount factor δ is taken as given in the bank's decision problem. In equilibrium, when the effect of G on δ, as given by equation (30) of Proposition 3, is taken into account, the relationship of the lending rate with the corn stock G is no longer monotonic or continuous, as we will find in the next subsection.

6 Equilibrium

In equilibrium, each party makes the rational (i.e. correct) expectation of other parties' decision. In particular, the discount factor δ used in the bank's decision on R^*, given in equation (45) of Proposition 4, satisfies equation (30) of Proposition 3; and the profile (D, R, r_2) used in investors' decision on δ, given in equation (30) of Proposition 3, is equal to $(D(R^*), R^*, r_2^*)$. We are interested in how the equilibrium lending rate R^e depends on the liquidity stock G. Suggested by Proposition 4, the equilibrium can be in five phases, depending on the range of G, with the range itself dependent on the equilibrium discount factor δ^e. In what follows, for each of these five phases, we find the range of G within which an equilibrium in the phase exists, and we characterize the equilibrium. Then, we put all the phases together and find how (R^e, δ^e) depends on the liquidity stock G in its full spectrum. Below for $k \in \{1, 2, 3, 4, 5\}$, we let δ_k and R_k denote respectively the value of δ^e and R^e in a Phase-k equilibrium.

Phase 1: In this phase, the bank is in Regime A: $G < \gamma D$. By equation (30), $\delta_1 = \delta_e$; and by equation (45), $R_1 = \frac{1}{\alpha}c_A(\delta_e) = 1/(\alpha\delta_e)$. By Proposition 4, a Phase-1 equilibrium exists if $G < \Gamma\left(\frac{1}{\alpha}c_A(\delta_1), \delta_1\right)$, which, given $\delta_1 = \delta_e$, is equivalent to $G < \Gamma\left(\frac{1}{\alpha} c_A(\delta_e), \delta_e\right)$. Altogether, we have

Claim 1 *If the liquidity stock* $G < \Gamma\left(\frac{1}{\alpha} c_A(\delta_e), \delta_e\right)$, *then there is a Phase-1 equilibrium, in which the bank is in Regime A and* $(R_1, \delta_1) = \left(\frac{1}{\alpha\delta_e}, \delta_e\right)$ *independent of* G.

Observe that because in Phase 1, the lemons problem is overcome by the inverse risk-shifting effect, the bank incurs no extra costs of borrowing liquidity and charges the same interest rate as in the case where no borrowing is needed: $R_1 = R^{NW}$, where R^{NW} is the lending rate when $\omega = 0$. The Phase-1 equilibrium is illustrated as follows.

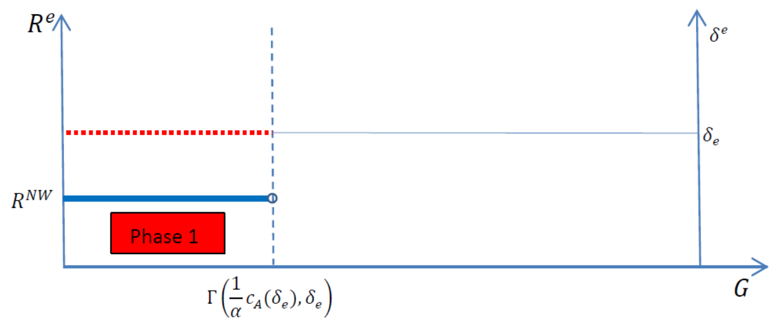

Figure 3: The Phase-1 Equilibrium, the lending rate R^e (the solid line) and discount factor δ^e (the dashed line) as a function of the liquidity stock G. In this equilibrium, the inverse risk-shifting effect overcomes the lemons problem.

Phase 2: In this phase, the bank is at the boundary of Regime B: $G = \gamma D$ and $R_2 = R^B(G, \delta_2)$. By Proposition 3, $\delta_2 = \overline{\delta_B}(r_2/R_2)$, determined by equation (29). In this phase, $r_2 = r^B$ and r^B is given by equation (42), a function of δ only. Substitute (42) for r_2 and we find in a Phase-2 equilibrium, investors' decision on the discount factor commands that $\delta_2 = \overline{\delta_B}\left(r^B(\delta_2)/R_2\right)$, from which explicitly we find

$$R_2 = \phi(\delta_2) := \frac{(1-q_L)(1-\omega)}{(1-x)\delta_2 + x\delta_L - q_L} \times \frac{(1-p)(\delta_H - \delta_2)}{\delta_e - \delta_2}, \tag{46}$$

with

$$x := (1-p)(1-q_H). \tag{47}$$

Lemma 11 $\phi' > 0$. *That is, in Phase 2, the lending rate is positively related to the discount factor.*

Proof. See Appendix A. ■

This positive relationship of R_2 with δ_2 results from investors' decision on the discounting factor, which is in contrast to the R-δ relationship resulting from the bank's decision, which is negative: The higher the discount factor, the lower the bank's funding cost and hence the lower the lending rate. The positive R-δ relationship in Lemma 11 is driven by $\partial\gamma/\partial R < 0$ given in Lemma 3. In Phase 2, each unit of lending is cushioned and serviced by $\gamma = G/D$ units of corn. If R rises, the quantity γ of corn to service a unit of lending decreases (as $\partial\gamma/\partial R < 0$) and the liquidity demand $\omega - \gamma$ that needs the external liquidity to meet increases, which pushes up the discount factor δ.

Equation (46) is independent of G. In Phase 2, thus, the corn stock G affects the discount factor only via its impact on the lending rate R_2, which is due to $R_2 = R^B(G, \delta_2)$, equivalent to $G = \Gamma(R_2, \delta_2)$. Using $R_2 = \phi(\delta_2)$,

$$G = \Gamma(\phi(\delta_2), \delta_2). \tag{48}$$

$[\Gamma(\phi(\delta), \delta)]'_\delta = \Gamma'_R\phi' + \Gamma'_\delta < 0$ because $\Gamma'_R < 0$ and $\Gamma'_\delta < 0$ by Lemma 10. Therefore, a Phase 2-equilibrium is unique. Moreover, in the equilibrium, with the liquidity stock G rising, the discount factor δ_2 falls, and so does the lending rate $R_2 = \phi(\delta_2)$. As we saw, the discount factor falls only because the lending rate falls. The latter is driven by Lemma 9, where we saw a rise in the liquidity stock G enlarges the lending scale and reduces the lending rate, by giving the bank a larger loss-absorbing capacity. The Phase-2 equilibrium thus goes the same way as one might *intuitively think*.

By Proposition 4, a Phase-2 equilibrium exists if $\Gamma\left(\frac{1}{\alpha}c_A(\delta_2), \delta_2\right) \leq G \leq \Gamma\left(\frac{1}{\alpha}c_{B1}(\delta_2), \delta_2\right)$, which, given equation (48) and $\Gamma'_R < 0$, is equivalent to

$$\frac{1}{\alpha}c_A(\delta_2) \geq \phi(\delta_2) \geq \frac{1}{\alpha}c_{B1}(\delta_2). \tag{49}$$

Both $c_A(\delta)$ (defined in 38) and $c_{B1}(\delta)$ (defined in 39) are a decreasing function. At $\delta = \delta_L$, $\frac{1}{\alpha}c_{B1} = \frac{1}{\alpha}c_A > 1 > \phi$. Obviously, if $\delta \to \delta_e$, $\phi \to \infty$ and is greater than $\frac{1}{\alpha}c_A$ and $\frac{1}{\alpha}c_{B1}$. Condition (49) is equivalent to $\overline{\delta_2} \geq \delta_2 \geq \underline{\delta_2}$, where $\overline{\delta_2}$ and $\underline{\delta_2}$ are respectively the unique root of $\frac{1}{\alpha}c_A(\delta) = \phi(\delta)$ and $\frac{1}{\alpha}c_{B1}(\delta) = \phi(\delta)$, as is illustrated as follows.

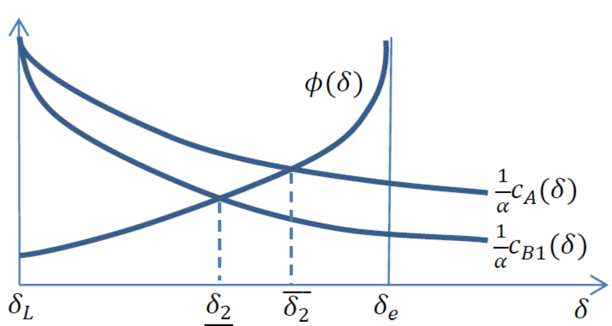

Figure 4: Condition (49) holds if and only if $\overline{\delta_2} \geq \delta_2 \geq \underline{\delta_2}$ and $\delta_e > \overline{\delta_2} > \underline{\delta_2} > \delta_L$.

Claim 2 *If* $G \in \left[\Gamma\left(\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right), \overline{\delta_2}\right), \Gamma\left(\frac{1}{\alpha}c_{B1}\left(\underline{\delta_2}\right), \underline{\delta_2}\right)\right]$, *then there is a unique Phase-2 equilibrium. If* G *ascends through this interval, the equilibrium discount factor* δ_2 *decreases from* $\overline{\delta_2}$ *to* $\underline{\delta_2}$ *and the equilibrium lending rate* $R_2 = \phi(\delta_2)$ *decreases from* $\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right)$ *to* $\frac{1}{\alpha}c_{B1}\left(\underline{\delta_2}\right)$.

Proof. See Appendix A. ■

We have seen the Phase-1 equilibrium exists if $G < \Gamma\left(\frac{1}{\alpha}c_A(\delta_e), \delta_e\right)$, where the lower bound of G for the Phase-2 equilibrium is $\Gamma\left(\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right), \overline{\delta_2}\right)$. The comparison between the two thresholds is facilitated by the following lemma.

Lemma 12 *If*

$$\frac{q_L(1-\delta_e)}{\delta_e - q_L}\omega(1-\omega) > \frac{1-\alpha}{\alpha}, \tag{50}$$

then $\Gamma\left(\frac{1}{\alpha}c_A(\delta), \delta\right)$ *is an increasing function of* δ *over* $[\delta_L, \delta_e]$; $\Gamma\left(\frac{1}{\alpha}c_A(\delta_L), \delta_L\right) > 0$; *and* $\frac{1}{\alpha}c_{B1}\left(\underline{\delta_2}\right) > R^{NW}$.

Proof. See Appendix A. ■

In what follows, we assume (50) holds true. As a result, $\Gamma\left(\frac{1}{\alpha}c_A(\delta_e), \delta_e\right) > \Gamma\left(\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right), \overline{\delta_2}\right) > 0$ and $\frac{1}{\alpha}c_{B1}\left(\underline{\delta_2}\right) > R^{NW}$. Then the equilibria of Phases 1 and 2 can be illustrated as follows.

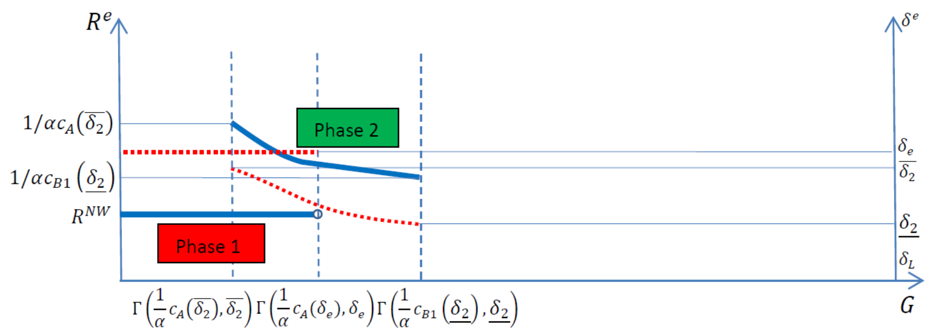

Figure 5: The equilibria of Phases 1 and 2, the lending rate R^e (the solid line) and discount factor δ^e (the dashed line). In Phase 2, a rise in the liquidity stock G, by giving the bank a greater loss-absorbing capacity, allows for a larger lending scale and a smaller lending rate, as one might intuitively think.

Phase 3: In this phase, on the one hand, $G > \gamma D$ and the bank is inside Regime B, while on the other hand, $G < \omega D$ and it needs the external liquidity to meet the liquidity demand. By Proposition 3, the discount factor δ_3 is determined by equation (31). By Proposition 4, the equilibrium lending rate $R_3 = \frac{1}{\alpha}c_{B1}(\delta_3)$. The two sides put together, δ_3 is determined by the following equation.

$$G = D\left(\frac{1}{\alpha}c_{B1}(\delta_3)\right)\left[\omega - \frac{p(\delta_3 - \delta_L)\delta_3}{(1-p)(\delta_H - \delta_3)}\frac{1}{\alpha}c_{B1}(\delta_3)\right] := \varphi(\delta_3). \tag{51}$$

And $R_3 = \frac{1}{\alpha} c_{B1}(\delta_3)$. Therefore, different to Phase 2, in Phase 3, the liquidity stock G affects the lending rate only via its impact on the discount factor. The following property of function $\varphi(\cdot)$ is important for the analysis.

Lemma 13 *There exists a $\delta_3^* \in [\delta_L, \delta_H)$ such that $\varphi' > 0$ for $\delta < \delta_3^*$ and $\varphi' < 0$ for $\delta > \delta_3^*$. $\varphi' < 0$ throughout $[\delta_L, \delta_H)$ if*

$$\frac{p}{1-p} \geq \frac{\alpha}{1-\alpha} \frac{\delta_H - \delta_L}{\delta_L}. \tag{52}$$

Proof. See Appendix A. ■

To simplify the exposition, we assume (52) and focus on the case in which $\varphi(\cdot)$ is decreasing throughout $[\delta_L, \delta_H)$. By Proposition 4, a Phase-3 equilibrium exists if $\Gamma\left(\frac{1}{\alpha} c_{B1}(\delta_3), \delta_3\right) \leq G \leq \omega D\left(\frac{1}{\alpha} c_{B1}(\delta_3)\right)$, which, with equation (51), is equivalent to

$$\Gamma\left(\frac{1}{\alpha} c_{B1}(\delta_3), \delta_3\right) \leq \varphi(\delta_3) \leq \omega D\left(\frac{1}{\alpha} c_{B1}(\delta_3)\right). \tag{53}$$

The second inequality holds for any $\delta_3 \in [\delta_L, \delta_H)$. The first one, with certain rearrangement, is equivalent to $\frac{1}{\alpha} c_{B1}(\delta_3) \geq \phi(\delta_3)$, which, by Figure 4, is equivalent to $\delta_3 \leq \underline{\delta_2}$. Therefore, inequalities of (53) hold if and only if $\delta_3 \in \left[\delta_L, \underline{\delta_2}\right]$, which, with equation (51), is equivalent to $G \in \left[\varphi\left(\underline{\delta_2}\right), \varphi(\delta_L)\right] = \left[\Gamma\left(\frac{1}{\alpha} c_{B1}\left(\underline{\delta_2}\right), \underline{\delta_2}\right), \omega D\left(\frac{1}{\alpha} c_{B1}(\delta_L)\right)\right]$. Hence, if G is within this interval, a Phase-3 equilibrium uniquely exist, as is illustrated as follows.

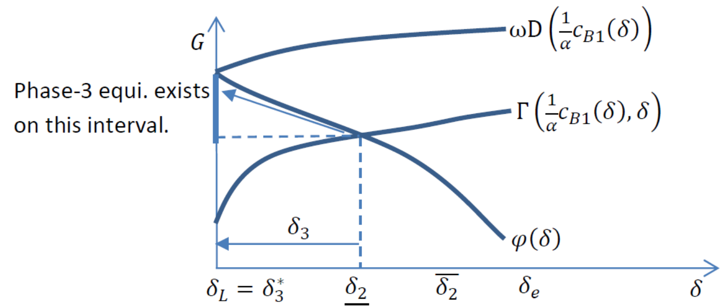

Figure 6: Given that $\varphi(\delta)$ is decreasing, if $G \in \left[\Gamma\left(\frac{1}{\alpha} c_{B1}\left(\underline{\delta_2}\right), \underline{\delta_2}\right), \omega D\left(\frac{1}{\alpha} c_{B1}(\delta_L)\right)\right]$ there is a unique Phase-3 equilibrium and the equilibrium discount factor δ_3 decreases with G.

Intuitively, the decreasing of δ_3 with the liquidity stock G is due to a mechanism parallel to that in Malherbe (2014). The larger the liquidity stock G, the smaller the bank's liquidity need $wD - G$ that is met the external liquidity; hence the worse the lemon's problem and the greater the discount investors apply to evaluate the loans. With δ_3 decreasing, the lending cost $c_{B1}(\delta_3)$ is increasing and so is the lending rate $R_3 = \frac{1}{\alpha} c_{B1}(\delta_3)$. Observe that this reduces the lending scale D and thus the genuine liquidity need $\omega D - G$ that is met with the external liquidity, causing the discount factor to fall even further.

To summarize:

Claim 3 *Assume (52) holds. If* $G \in \left[\Gamma\left(\frac{1}{\alpha}c_{B1}\left(\underline{\delta_2}\right), \underline{\delta_2}\right), \omega D\left(\frac{1}{\alpha}c_{B1}\left(\delta_L\right)\right)\right]$, *there exists a unique Phase-3 equilibrium in which the bank is within Regime B. If G ascends through the interval, the equilibrium discount factor* δ_3 *decreases from* $\underline{\delta_2}$ *to* δ_L, *and the equilibrium lending rate* $R_3 = \frac{1}{\alpha}c_{B1}(\delta_3)$ *increases from* $\frac{1}{\alpha}c_{B1}\left(\underline{\delta_2}\right)$ *to* $\frac{1}{\alpha}c_{B1}(\delta_L)$.

Built on Figure 5, the equilibria of Phases 1 to 3 can be illustrated as follows.

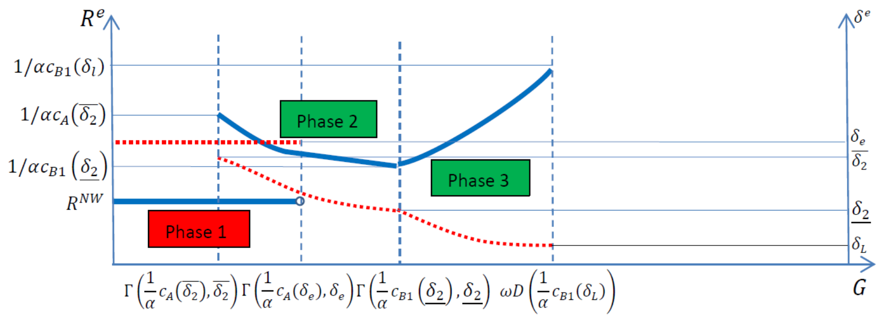

Figure 7: The equilibria of Phases 1 to 3, the lending rate R^e (the solid line) and discount factor δ^e (the dashed line). In Phase 3, in the mechanism of Malherbe (2014), a rise in the liquidity stock G aggravates the lemons problem, causing the discount factor δ_3 to fall and thus the lending rate R_3 to rise.

Phase 4: In this phase, the bank is at the boundary of Scenario B2, in which it can self-satisfy its liquidity needs: $G = \omega D$. This equation pins down the bank's lending rate: $R_4 = \overline{R}(G) = \overline{A}\alpha(\omega/G)^{1-\alpha}$. By Proposition 3, $\delta_4 = \delta_L$; as the bank needs no external liquidity to meet the liquidity demand, investors regard any offer of exchanging its loans with liquidity as an attempt to dump lemons. By Proposition 4, a Phase-4 equilibrium exists if $G \in \left[\omega D\left(\frac{1}{\alpha}c_{B1}(\delta_4)\right), \omega D\left(\frac{1}{\alpha}c_{B2}(\delta_4)\right)\right] = \left[\omega D\left(\frac{1}{\alpha}c_{B1}(\delta_L)\right), \omega D\left(\frac{1}{\alpha}c_{B2}(\delta_L)\right)\right]$. Observe that $\frac{1}{\alpha}c_{B2}(\delta_L) = 1/(\alpha\delta_e) = R^{NW}$. To summarize,

Claim 4 *If* $G \in \left[\omega D\left(\frac{1}{\alpha}c_{B1}(\delta_L)\right), \omega D\left(\frac{1}{\alpha}c_{B2}(\delta_L)\right)\right]$, *there is a unique Phase-4 equilibrium, in which* $\delta_4 = \delta_L$, *and* $R_4 = \overline{A}\alpha(\omega/G)^{1-\alpha}$. *If G ascends through the interval, the equilibrium lending rate* R_4 *decreases from* $\frac{1}{\alpha}c_{B1}(\delta_L)$ *to* $\frac{1}{\alpha}c_{B2}(\delta_L) = R^{NW}$.

This decreasing relationship is because in Phase 4, self-satisfaction of the liquidity needs imposes a binding constraint $\omega D(R) \leq G$ is binding. Therefore, a greater liquidity stock G, by giving the bank a larger capacity to meet its liquidity needs, relaxes the constraint, hence allowing for a higher lending scale and a lower lending rate. Phase 4 thus follows the intuitive thinking, as in Phase 2.

Built on Figure 7, the equilibria of Phases 1 to 4 can be illustrated as follows.

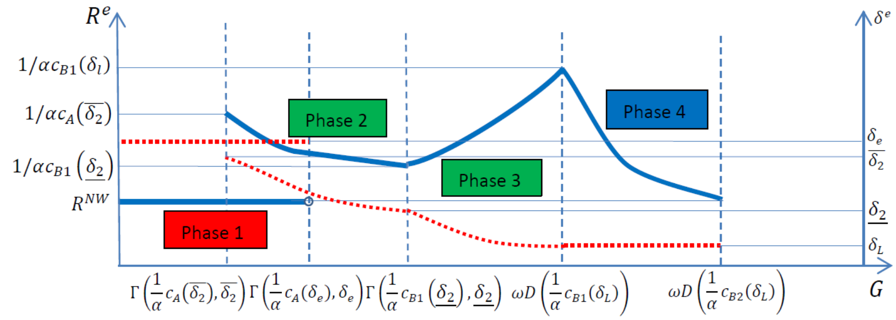

Figure 8: The equilibria of Phases 1 to 4, the lending rate R^e (the solid line) and discount factor δ^e (the dashed line). In Phase 4, the lending rate decreases the liquidity stock G because a larger stock give the bank a larger capacity to meet its liquidity needs, as the intuitive thinking would suggest.

Phase5: Lastly, in this phase, the bank is inside the self-satisfaction case: $G > \omega D$. As in Phase 4, any attempt of exchanging its loans with liquidity is interpreted by investors as dumping lemons and thus $\delta_5 = \delta_L$. However, different to Phase 4, now the self-satisfaction constraint is no longer binding. As a result, a larger liquidity stock G relaxes the constraint no longer, nor affects the lending rate. By Proposition 4 $R_5 = \frac{1}{\alpha}c_{B2}(\delta_L) = R^{NW}$, independent of G, and the Phase-5 equilibrium exists if $G \geq \omega D\left(\frac{1}{\alpha}c_{B2}(\delta_5)\right) = \omega D\left(\frac{1}{\alpha}c_{B2}(\delta_L)\right)$. To summarize,

Claim 5 *If $G \geq \omega D\left(\frac{1}{\alpha}c_{B2}(\delta_L)\right)$, there is a unique Phase-5 equilibrium, in which $\delta_5 = \delta_L$ and $R_5 = R^{NW}$ independent of G.*

Observe that in the Phase-5 equilibrium, as in the Phase-1 equilibrium, the lending rate is equal to R^{NW}, the one that the bank charges when it faces no noisy withdrawal. However, the reasons are different. In Phase 1, the equality is because $\delta^e = \delta_e$ that is, the lemons' problem is overcome and incurs no extra costs of liquidity borrowing. In Phase 5, it is because the bank's liquidity stock abounds exempting the bank from the need of liquidity borrowing.

With all of the five phases examined, we finally reach the full picture on the effect of the bank's liquidity stock G on its equilibrium lending rate R^e and discount factor δ^e under Assumptions (50) and (52). Adding the Phase 5 equilibrium to Figure 8, this full picture can be illustrated as follows.

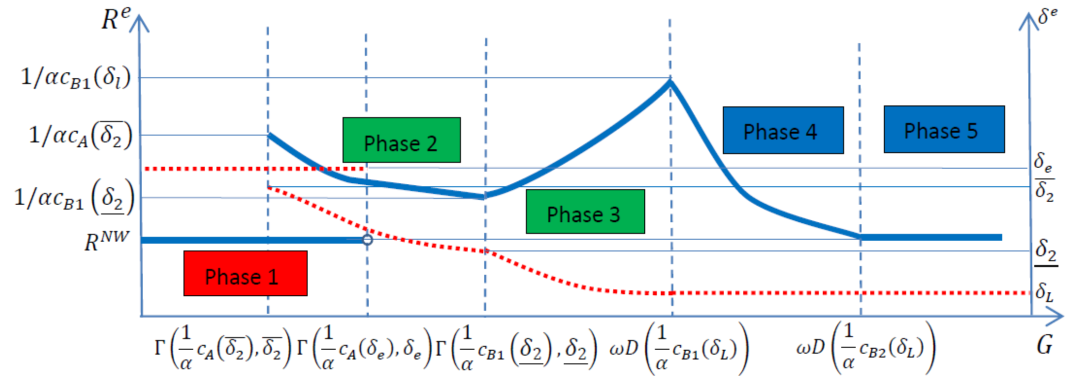

Figure 9: The full picture of the lending rate R^e (the solid line) and discount factor δ^e (the dashed line) in relation to the corn stock G. In Phase 1, $R^e = R^{NW}$ because the lemons problem is overcome by the inverse risk-shifting effect and incurs no extra costs for liquidity borrowing. In Phases 2 and 4, R^e decreases with G because a larger G, by giving the bank a greater capacity to absorb risks and to meet liquidity needs, allows for a larger lending scale, as one might intuitively think. In Phase 3, R^e increases with G in the in the mechanism of Malherbe (2014). In Phase 5, $R^e = R^{NW}$ because the liquidity stock abounds exempting the bank from the need of liquidity borrowing.

As depicted above, the equilibrium lending rate R^e has a discontinuous, non-monotonic relationship with the liquidity stock G. The discontinuity part is due to the regime switch, which occurs either at $G = \Gamma\left(\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right), \overline{\delta_2}\right)$ or at $G = \Gamma\left(\frac{1}{\alpha}c_A\left(\delta_e\right), \delta_e\right)$. In Regime A, the lemons problem is overcome by the inverse risk-shifting effect and incurs no extra costs for the bank to obtain the external liquidity, which is represented by the equilibrium discount factor is equal to δ_e, the discount factor that obtains if the bank has no private information about the loan quality. By contrast, in Regime B, the lemons problem is not overcome and adds costs for the bank to obtain the external liquidity, causing the equilibrium discount factor falling below δ_e. A switch between these two regimes, therefore, entails an abrupt change in the bank's funding cost, which induces a discontinuous change in its lending rate.

Recall that the liquidity stock actually acts in the identity of the safe asset when its rise beyond the thresholds forces the regime change from A to B: The smaller the safe asset, the greater the loss borne by the bank's debt, and the stronger the inverse risk-shifting effect. The link between the loss borne by the debt and the strength of the inverse risk-shifting effect drives a complementarity between the lending scale and the discount factor and this complementarity drives the existence of two equilibria when $G \in \left[\Gamma\left(\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right), \overline{\delta_2}\right), \Gamma\left(\frac{1}{\alpha}c_A\left(\delta_e\right), \delta_e\right)\right)$, one in Regime A, the other in Regime B, as we see from Figure 9. Specifically, if the bank expects a high discount factor, i.e. $\delta = \delta_e$, then it charges a low interest $1/\left(\alpha\delta_e\right)$; as a result, its lending scale D is large and thus the bad-state loss borne by the debt is high, which induces an inverse risk-shifting effect strong enough to overcome the lemons problem, and hence the bank is indeed in

Regime A and the discount factor is indeed δ_e. By a parallel argument, if the bank expects a low discount factor, i.e. $\delta = \delta_2$, then it will end up in Regime B and this expectation is also self-fulfilled.

Within Regime B, which can be in equilibrium if $G \geq \Gamma\left(\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right), \overline{\delta_2}\right)$, both what the intuitive thinking would predict and what Malherbe (2014) would predict are a part of the full picture. In Phase 2, the lending is at the border of Regime B, and in Phase 4, it is at that of the scenario in which its liquidity needs are self-satisfied. Namely, the border constraint – $G \geq \gamma D$ for Phase 2 and $G \geq \omega D$ for Phase 4 – is binding. Hence, a higher liquidity stock G relaxes these constraints and allows for a larger lending scale D and hence a lower lending rate. This prediction conforms with what one might intuitively think. On the other hand, in Phase 3, where $\gamma D < G < \omega D$ in equilibrium, that is, the bank is inside Regime B and far away from the self-satisfaction scenario, this effect of relaxing a binding constraint does not obtain. Rather, a higher stock G of the bank's own liquidity means a smaller genuine liquidity need that is to be met with the external liquidity and a greater chance of lemon dumping, hence, a lower discount factor, a higher funding cost and thus a higher lending rate. This prediction conforms with Malherbe (2014).

As we have seen, the liquidity stock G acts in the identity of the liquid asset in Phase 3 and in the identity of the safe asset when its change enforces the regime switch between Phases 1 and 2. The double identity of the liquidity stock hence plays an important part in its relationship with the bank's lending rate and lending scale.

6.1 Discussion of the assumptions

The full picture is built on the four assumptions that we have made: (4), (16), (50) and (52). Of these assumptions, the first two are the ones that drive the results. Assumption (4) ensures that bank borrowers default in the contingency of $\widetilde{A} = \underline{A}$. If they never default, then each loan will be worth its face value and the bank will have no private information about the loan quality, namely, the lemons problem will be absent. Assumption (16) ensures the date-1 liquidity demand is large enough, which is necessary to drive the discontinuous, non-monotonic effect of the liquidity stock on the bank's lending behaviour, as we have seen if the demand is zero then it has no impact on the latter. On the other hand, Assumptions (50) and (52) are not essential for our results. They are made solely to simplify the exposition. Assumption (50) ensures that $\Gamma\left(\frac{1}{\alpha}c_A\left(\delta_e\right), \delta_e\right) > \Gamma\left(\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right), \overline{\delta_2}\right)$, that is, there is no gap for the value of G between Phase 1 and 2. If such a gap exists, then for $G \in \left[\Gamma\left(\frac{1}{\alpha}c_A\left(\delta_e\right), \delta_e\right), \Gamma\left(\frac{1}{\alpha}c_A\left(\overline{\delta_2}\right), \overline{\delta_2}\right)\right]$, in equilibrium the bank is to play a mixed strategy with X_L at date 1 and accordingly the discount factor that investors use at date 1 is δ_M, as is illustrated in Figure 1. This complicates the picture, but will not change it qualitatively. Assumption (52) lets us focus on the case in which function $\varphi(\cdot)$ defined in (51) is decreasing over $[\delta_L, \delta_H)$. If $\varphi(\cdot)$ is not so, by Lemma 13, $\varphi(\cdot)$ is in a "$\wedge$" shape and the case is analyzed in Appendix B, where we show there are two

Phase-3 equilibria, one as we have studies, the other being unstable in the sense of Malherbe (2014). This, again, complicates the picture, but will not qualitatively change it.

Now we show that a non-empty area in the parameter space is demarcated by these four assumptions altogether. Assumption (4) follows from

$$\frac{\underline{A}}{\overline{A}\alpha} < \frac{q_L}{1-\alpha(1-q_L)}, \tag{54}$$

while the other three assumptions are replicated below:

$$\begin{aligned} \omega &\geq 1-\frac{\underline{A}}{\overline{A}\alpha} & (55)\\ \frac{p}{1-p} &\geq \frac{\alpha}{1-\alpha}\frac{\delta_H-\delta_L}{\delta_L} & (56)\\ \frac{q_L(1-\delta_e)}{\delta_e-q_L}\omega(1-\omega) &> \frac{1-\alpha}{\alpha}. & (57) \end{aligned}$$

Given (q_H, q_L) that satisfies $1 > q_H > q_L > 0$ and $\omega \in (0,1)$, we can pick a $\underline{A}/\left(\overline{A}\alpha\right)$ close to 1 enough to satisfy (55), and then an α close to 1 enough to satisfy (54) and (57) at $p = 1/2$, and lastly we pick a $p > 1/2$ close to 1 enough to meet (56). Observe that the left hand side of (57) decreases with $\delta_e = p\delta_L + (1-p)\delta_H$ and hence increases with p. Thus if it is satisfied at $p = 1/2$, then it is also satisfied for any $p > 1/2$. Hence, for any given profile $(q_H, q_L, \omega) \in (0,1)^3$ such that $q_H > q_L$, there is a non-empty set of $\left(\underline{A}/\left(\overline{A}\alpha\right), \alpha, p\right)$ that satisfies all the four inequalities from (54) to (57).

7 Conclusion

In response to the 2008 financial crisis, the Basel Committee introduces the Liquidity Coverage Ratio standard. This paper considers how a bank's stock of liquid assets affect its lending rate and lending scale, when the market for external liquidity is beset with the lemons problem. We find that the relationship of the liquidity stock with the lending behaviour is discontinuous and non-monotonic. A crucial part for this relationship is played by the inverse risk-shifting effect, which is due to the double identity of liquid assets. Namely, these assets are typically also safe assets. Hence, exchanging lemon assets with external liquidity (in the means of either asset sale or collateralized borrowing) amounts to inverse risk-shifting, which reduces the equity value as the seminal work of Jensen and Meckling (1976) shows. This inverse risk-shifting effect therefore provides the bank with disincentives to dump lemons. Indeed, when the liquidity stock is below a threshold, the bank's safe assets are meagre and the inverse risk-shifting effect is strong enough to overcome the lemons problem, which thus adds no costs for the bank to obtain external liquidity. It starts adding to the cost of external liquidity if the liquidity stock is above the threshold. Therefore, when the stock ascends above the threshold, the bank's funding cost abruptly rises and hence so does its lending rate, and consequently its lending scale abruptly falls.

This fact show that a larger liquidity stock is not always a bless. Another reason for it become a curse is the mechanism found by Malherbe (2014), when the stock is within a certain interval. However, there are intervals if the stock is within which, a larger liquidity stock is a bless – namely leading to a lower lending rate and higher lending scale. It generates these effects by enlarging the bank's capacity to absorb risks or to manage its liquidity demand, as one might intuitively think.

References

[1] Akerlof, G. (1970). The Market for "Lemons": Quality Uncertainty and the Market Mechanism. *Quarterly Journal of Economics*, 84, 488-500.

[2] Bianchi, J., and S. Bigio (2017). Banks, liquidity management and monetary policy. Federal Reserve Bank of Minneapolis Staff Report 503.

[3] Bigio, Saki (2015). Endogenous liquidity and the business cycle, *American Economic Review* 105 (6): 1883–1927.

[4] Bolton, P., T. Santos and J. Scheinkman, (2011), Outside and inside liquidity, *Quarterly Journal of Economics,* 126, 259–321.

[5] Bond, P. and Y. Leitner (2015). "Marketrun-ups, marketfreezes, inventories, and leverage". Journal ofFinancialEconomics, 115, 155–167.

[6] Diamond, D. W., and P. H. Dybvig (1983). Bank Runs, Deposit Insurance, and Liquidity. *Journal of Political Economy,* 91, 401-419.

[7] Donaldson, J. D., G. Piacentino, and A. Thakor (2018). Warehouse Banking. *Journal of Financial Economics*, 129, 250–67.

[8] Eisfeldt, A. L. (2004). "Endogenous Liquidity in Asset Markets." Journal of Finance 59 (1): 1–30.

[9] Faure, S. and H. Gersbach (2016). Money Creation and Destruction. CFS Working Paper No. 555.

[10] Gomez, F. and Q. Vo (2020). "Liquidity management, fire sale and liquidity crises in banking: the role of leverage". Staff Working Paper No. 894, Bank of England.

[11] Heider, F., M. Hoerova and C. Holthausen (2015). Liquidity hoarding and interbank market rates: The role of counterparty risk. *Journal of Financial Economics*, 118, 336-354.

[12] Jakab, Zoltan and Michael Kumhof (2015). Banks are not intermediaries of loanable funds - and why this matters. Bank of England Working Paper No. 529.

[13] Jensen, Michael C. and William H. Meckling (1976). Theory of the Firm: Managerial Behavior, Agency Costs and Ownership Structure. *Journal of Financial Economics*, 3(4), 305-60.

[14] Kirabaeva, K. (2011). Adverse Selection, Liquidity, and Market Freezes. Unpublished.

[15] Kurlat, P. (2013). "Lemons Markets and the Transmission of Aggregate Shocks." *American Economic Review* 103 (4): 1463–89.

[16] Kurlat, P. (2018). "Liquidity as Social Expertise," Journal of Finance, American Finance Association, 73(2), 619-656.

[17] Mendizábal, H. R., Narrow Banking with Modern Depository Institutions: Is there a Reason to Panic? forthcoming, *International Journal of Central Banking.*

[18] Malherbe, F. (2014). Self-fulfilling Liquiditydry-Ups. *Journal of Finance,* 69, 947–970.

[19] Morrison, A. and T. Wang (2019). Bank liquidity, bank lending, and "bad bank" policies. https://core.ac.uk/download/373373584.pdf

[20] Parlour, C. A. and G. Plantin (2008), Loan sales and relationship banking, *Journal of Finance,* 63, 1291–1314.

[21] Wang, T. (2019). Banks' Wealth, Banks' Creation of Money, and Central Banking. *International Journal of Central Banking,*15, 89-135.

[22] Wang, T. (2020). An Interbank Network Structured by the Real Economy. http://repository.essex.ac.uk/30021/

[23] Wang, T. (2021). The Liquidity Constraint of Banks and Monetary Non-Neutrality in the Steady State. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3857062

Appendix A: Proofs

The proof of Lemma 1

Proof. First, we prove that under Assumption (4), entrepreneurs default in the bad state with any lending interest R. Facing lending rate R, their decision problem on the number D units of deposit to borrow is given by (3). If they do not default in the bad state, this decision problem becomes:

$$\max_D \left[q_e \overline{A} + (1-q_e)\underline{A}\right] D^\alpha - DR.$$

At the optimum,

$$D = \left(\frac{\left[q_e \overline{A} + (1-q_e)\underline{A}\right]\alpha}{R}\right)^{\frac{1}{1-\alpha}}.$$

Then, in the bad state: $\underline{A}D^a = \underline{A}D^{a-1} \times D = DR \times \frac{\underline{A}}{\left[q_e \overline{A} + (1-q_e)\underline{A}\right]\alpha} < DR$ under Assumption (4), which contradicts with the supposition that they do not default in the bad state.

Given they default, entrepreneurs' decision problem becomes:

$$\max_D \overline{A}D^\alpha - DR.$$

At the optimum, their demand of bank deposit is thus given by (5). In the bad state, as they default, all their output is passed on to the bank. Hence the repayment to the bank is $\underline{A}D^a = \underline{A}D^{a-1} \times D = \frac{\underline{A}}{\overline{A}\alpha} \times DR$.

■

The proof of Lemma 3

Proof. At $\widetilde{q} = q_L$, from (22) it follows that $V'(X) > 0$ if $\widetilde{V}_B(X) > 0$ and $V'(X) < 0$ if $\widetilde{V}_B(X) < 0$. Furthermore, by (18), $\widetilde{V}'_B(X) = -\frac{\underline{A}/(\overline{A}\alpha)}{\delta} + 1 > 0$ because $\underline{A}/\left(\overline{A}\alpha\right) < q_L + (1-q_L)\underline{A}/\left(\overline{A}\alpha\right) = \delta_L < \delta$. Hence, if $\widetilde{V}_B(\underline{X}) \geq 0$, then all $X \in (\underline{X}, \overline{X}]$, $\widetilde{V}_B(X) > 0$ and hence $V'(x) > 0$ and $X_L = \overline{X}$; and if $\widetilde{V}_B\left(\overline{X}\right) \leq 0$, then $V'(X) < 0$ always and $X_L = \underline{X}$. Let G_1 and G_2 be the value of G defined by the following equations:

$$\begin{aligned} \widetilde{V}_B(\underline{X}, G_1) &= 0 \\ \widetilde{V}_B\left(\overline{X}, G_2\right) &= 0. \end{aligned}$$

Then, $G_1 > G_2$ because $\widetilde{V}_B$ strictly increases with G by (18). If $G \geq G_1$, then $\widetilde{V}_B(\underline{X}) \geq 0$ and hence $X_L = \overline{X}$; and if $G \leq G_2$, then $\widetilde{V}_B\left(\overline{X}\right) \leq 0$ and hence $X_L = \underline{X}$. For $G \in (G_2, G_1)$, then $\widetilde{V}_B(\underline{X}) < 0$ and $\widetilde{V}_B\left(\overline{X}\right) > 0$. For such a G, the graph of $V(X)$ is in a "V" shape and hence X_L is either $\overline{X}$ or $\underline{X}$ depends on the comparison of $V\left(\overline{X}\right)$ to $V(\underline{X})$. Observe that $V\left(\overline{X}\right) = q_L V_G\left(\overline{X}\right) + (1-q_L)\widetilde{V}_B\left(\overline{X}\right) = DR\delta + G - \omega D - (1-\omega)Dr_2$ and

$$\begin{aligned} V(\underline{X}) &= q_L V_G(\underline{X}) \\ &= \left\{ \begin{array}{c} q_L\left[DR - (\omega D - G)/\delta - (1-\omega)Dr_2\right] \text{ if } \underline{X} = \omega D - G \geq 0 \\ q_L\left[DR - (\omega D - G) - (1-\omega)Dr_2\right] \text{ if } \omega D - G < 0 \end{array} \right\}. \end{aligned}$$

Observe that $\left(V\left(\overline{X}\right) - V(\underline{X})\right)'_G \geq 1 - (q_L/\delta) > 0$ because $q_L < q_L + (1-q_L)\underline{A}/\left(\overline{A}\alpha\right) = \delta_L < \delta$. Recall that at $G = G_2$, $V\left(\overline{X}\right) - V(\underline{X}) < 0$ and at $G = G_1$, $V\left(\overline{X}\right) - V(\underline{X}) > 0$. Hence, there exists $G^* \in (G_2, G_1)$ such that if $G < G^*$, then $V\left(\overline{X}\right) - V(\underline{X}) < 0$ and hence $X_L = \underline{X}$; and if $G > G^*$, then $X_L = \overline{X}$.

This G^* is determined by $V\left(\overline{X}\right) - V\left(\underline{X}\right) = 0$, which, if $\underline{X} = \omega D - G \geq 0$, is equivalent to

$$\begin{aligned}
DR\delta + G^* - \omega D - (1-\omega)\, Dr_2 &= q_L\left[DR - (\omega D - G^*)/\delta - (1-\omega)\, Dr_2\right] \Leftrightarrow \\
\left[1 - (q_L/\delta)\right] G^* &= \left[q_L R - R\delta + (1 - (q_L/\delta))\,\omega + (1-q_L)(1-\omega)\, r_2\right] D \Leftrightarrow \\
G^* &= \left[\frac{-(1-(q_L/\delta))\, R\delta}{1-(q_L/\delta)} + \omega + \frac{(1-q_L)(1-\omega)}{1-(q_L/\delta)} r_2\right] D \Leftrightarrow \\
G^* &= \left(\omega - \left(R - \frac{(1-q_L)(1-\omega)}{\delta - q_L} r_2\right)\delta\right) D \\
&= \gamma D.
\end{aligned}$$

Hence, (23) holds true. Observe that by lending out one unit of liability, the bank promises to pay r_2 and obtains at most R at date 2. The bank will always make sure that $R > r_2$. Also, by (8), $\delta \geq \delta_L = q_L + (1-q_L)\,\underline{A}/\left(\overline{A}\alpha\right)$. Then under Assumption (16), the following condition holds for any $\delta \geq \delta_L$:

$$R - \frac{(1-q_L)(1-\omega)}{\delta - q_L} r_2 > 0. \tag{58}$$

As a result, $\gamma = \omega - \left(R - \frac{(1-q_L)(1-\omega)}{\delta - q_L} r_2\right)\delta < \omega$ and $\partial\gamma/\partial\delta < 0$.

If $\omega D - G < 0$ and hence $\underline{X} = 0$, then $V\left(\overline{X}\right) - V\left(\underline{X}\right) = 0$ is equivalent to

$$\begin{aligned}
DR\delta + G - \omega D - (1-\omega)\, Dr_2 &= q_L\left[DR - (\omega D - G) - (1-\omega)\, Dr_2\right] \Leftrightarrow \\
G &= \left(\omega - \left(R - \frac{(1-q_L)(1-\omega)}{\delta - q_L} r_2\right)\frac{\delta - q_L}{1-q_L}\right) D \\
&= \gamma' D.
\end{aligned}$$

Observe that under condition (58), $\gamma' < \omega$ and also $\gamma < \omega$. Hence, if $\omega D - G < 0$, then $G > \omega D > \gamma' D$ and therefore $X_L = \overline{X}$, which conforms with equation (23).

Regarding the property of γ, that $\partial\gamma/\partial R = -\delta < 0$ is obvious, while we have shown above $\partial\gamma/\partial\delta < 0$ and $\gamma < \omega$.

As for the last claim, because $G^* \in (G_2, G_1)$, we have $\widetilde{V}_B\left(\underline{X}\right) < 0$ and $\widetilde{V}_B\left(\overline{X}\right) > 0$ at $G = G^* = \gamma D$. If $X_L = \overline{X}$, then by (23), $G \geq G^*$ and $\widetilde{V}_B(X_L) = \widetilde{V}_B\left(\overline{X}; G\right) \geq \widetilde{V}_B\left(\overline{X}; G^*\right) > 0$. If $X_L = \underline{X}$, then by (23), $G \leq G^*$ and $\widetilde{V}_B(X_L) = \widetilde{V}_B\left(\underline{X}; G\right) \leq \widetilde{V}_B\left(\underline{X}; G^*\right) < 0$. ■

Proof of Lemma 4

Proof. Let r^A denote the value of the threshold r_2^* if X is in Regime A, that is, $X_L = \underline{X} = \max(\omega D - G, 0)$. From the binding liquidity constraint (27), r^A is determined by

$$q_e r^A + (1-q_e)\min\left(r^A, \underline{r}(R, \delta, G)\right) = 1, \tag{59}$$

where from (19),

$$\begin{aligned}\underline{r}(R,\delta,G) \quad &:= \frac{\left(DR-\frac{\underline{X}}{\delta}\right)\frac{\underline{A}}{\overline{A}\alpha}+\underline{X}+G-\omega D}{(1-\omega)D} \\ &= \frac{\underline{A}/\left(\overline{A}\alpha\right)}{1-\omega}\left(R+\frac{1}{\delta}\min\left(G\left(\frac{R}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}}-\omega,0\right)+\frac{\overline{A}\alpha}{\underline{A}}\max\left(G\left(\frac{R}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}}-\omega,0\right)\right). \quad (60)\end{aligned}$$

Obviously, $\underline{r}(R,\delta,G)$ increases with all of its arguments. By (59), hence, r^A is a function decreasing with (R,δ,G), denoted by $r^A(R,\delta,G)$. ■

Let r^A denote the value of the threshold r_2^* in Regime B, that is, if $X_L=\overline{X}$. By Lemma 3, $\widetilde{V}_B\left(\overline{X}\right)>0$. Hence $\underline{r_2}(X_L)=r_2$ by (20). Then, by (25) and (27), r^B is determined by

$$[1-(1-p)(1-q_H)]\,r^B+(1-p)(1-q_H)\min\left(r^B,\underline{r}(R,\delta,G)\right)=1. \tag{61}$$

Because $\underline{r}(R,\delta,G)$ increases with (R,δ,G), $r^B(R,\delta,G)$ also decreases with all its arguments.

We are left to prove $r^A>r^B$. For this purpose, let function $r=f(x)$ be implicitly defined by

$$xr+(1-x)\min(r,\underline{r}(R,\delta,G))=1.$$

Then $r^A=f(q_e)$ and $r^B=f(1-(1-p)(1-q_H))=f(q_e+p(1-q_L))$. By the implicit function theorem, $f'(x)\le 0$. Hence, $r^A=f(q_e)\ge f(q_e+p(1-q_L))=r^B$.

Proof of Lemma 5

Proof. Let

$$\begin{aligned} a \quad &:= q_L+(1-q_L)(1-\omega)\frac{r_2}{R} \\ f(\delta,a) \quad &:= \frac{p}{1-p}\frac{(\delta-\delta_L)(\delta-q_L)}{\delta_H-\delta}-\delta+a. \end{aligned}$$

Then equation (29) has a unique solution within (δ_L,δ_e) if and only if $f(\delta,a)$ has a unique root within (δ_L,δ_e). First, there exists one root, because $f(\delta_L,a)=-\left(\delta_L-q_L-(1-q_L)(1-\omega)\frac{r_2}{R}\right)=-(1-q_L)\left(\frac{\underline{A}}{\overline{A}\alpha}-(1-\omega)\frac{r_2}{R}\right)|_{r_2<R}<-(1-q_L)\left(\frac{\underline{A}}{\overline{A}\alpha}-(1-\omega)\right)|_{\text{assumption (16)}}<0$; and, given that $\frac{p}{1-p}\frac{\delta_e-\delta_L}{\delta_H-\delta_e}=1$, $f(\delta_e,a)=-q_L+a=(1-q_L)(1-\omega)\frac{r_2}{R}>0$. Second, we prove that the root is unique. For this purpose let δ_s denote the smallest of the roots. By definition, then for any $x<\delta_s$, $f(x,a)<0$ and $f(\delta_s,a)=0$. It follows that $f'_\delta(\delta_s,a)\ge 0$. Observe that $f(\cdot,a)$ is strictly convex over $[\delta_L,\delta_e]$. Hence, $f'_\delta(x)>f'_\delta(\delta_s)\ge 0$. Then for any $x>\delta_s$, $f(x,a)>f(\delta_s,a)=0$. That is, there is no root that is bigger than δ_s. Hence, δ_s is the unique root.

To prove that $\overline{\delta_B}'\left(\frac{r_2}{R}\right)<0$, it suffices to show that the unique root of $f(\delta,a)=0$ increases with a. For this purpose, denote the root by $\delta(a)$ and consider $a>a'$. Then $0=f(\delta(a),a)>f(\delta(a),a')$. Because $\delta(a')$ is the unique root of $f(\delta,a')=0$, the argument above shows that $f(x,a')<0$ only if $x<\delta(a')$. Therefore that $f(\delta(a),a')<0$ implies that $\delta(a)<\delta(a')$. ■

Proof of Proposition 3

Proof. Before we prove equation (30) branch by branch, we prove that $\delta \leq \delta_e$ always. That is because $X_H = \underline{X}$ and $X_L \geq \underline{X}$ and thus $X_L \geq X_H$; and then $\delta \leq \delta_e$ follows from (8).

We have seen that if $G \geq \omega D$, then $\underline{X} = 0$ and hence $X_H = \underline{X} = 0$ and by (8), $\delta = \delta_L$; that is, we have proven the last branch of (30). We thus only need to consider the case in which $G < \omega D$. In this case $X_H = \underline{X} > 0$ and hence by (8), $\delta > \delta_L$ and Lemma 3 can be applied.

When G starts falling from $G = \omega D$, because $\gamma < \omega$ due to condition (58), the bank enters Regime B in which $G \geq \gamma(\delta) D$. In this regime, by Lemma 3 $X_L = \overline{X} = DR\delta$, while $X_H = \underline{X} = \omega D - G$ Hence, by (8),

$$\begin{aligned}
\delta &= \frac{p\overline{X}}{p\overline{X} + (1-p)\underline{X}}\delta_L + \frac{(1-p)\underline{X}}{p\overline{X} + (1-p)\underline{X}}\delta_H \Leftrightarrow \\
\frac{p\overline{X}}{p\overline{X} + (1-p)\underline{X}}(\delta - \delta_L) &= \frac{(1-p)\underline{X}}{p\overline{X} + (1-p)\underline{X}}(\delta_H - \delta) \Leftrightarrow \\
\frac{\delta - \delta_L}{\delta_H - \delta} &= \frac{1-p}{p}\frac{\underline{X}}{\overline{X}} = \frac{1-p}{p}\frac{\omega D - G}{DR\delta},
\end{aligned}$$

which leads to

$$\frac{\delta - \delta_L}{\delta_H - \delta}\delta = \frac{1-p}{p}\frac{\omega D - G}{DR}, \tag{62}$$

that is, equation (31). Hence, $\delta = \delta_B$. Obviously, at $G = \omega D$, $\delta_B = \delta_L$ and hence δ is continuous with G at $G = \omega D$. Moreover, the left hand side of (31) increases with δ, while its right hand side decreases with G. Hence, δ_B decreases with G in Regime B, where $G \geq \gamma(\delta) D$.

With G falling from $G = \omega D$, the bank stays within Regime B until G reaches the boundary $G = \gamma(\delta) D$, substituting which into (62), which then becomes

$$\begin{aligned}
\frac{\delta - \delta_L}{\delta_H - \delta}\delta &= \frac{1-p}{p}\frac{\omega D - \gamma(\delta) D}{DR} \\
|_{(24)} &= \frac{1-p}{p}\frac{\left(R - \frac{(1-q_L)(1-\omega)}{\delta - q_L}r_2\right)\delta}{R},
\end{aligned}$$

which is equivalent to

$$\frac{\delta - \delta_L}{\delta_H - \delta} = \frac{1-p}{p}\left[1 - \frac{(1-q_L)(1-\omega)}{\delta - q_L}\frac{r_2}{R}\right],$$

that is, equation (29). By Lemma 5, it has a unique solution for δ, which is $\overline{\delta_B}$. Therefore, the lower bound of G for the bank to stay in Regime B is $G = \gamma\left(\overline{\delta_B}\right) D$, while the upper bound is ωD. It follows that if $G \in \left[\gamma\left(\overline{\delta_B}\right) D, \omega D\right]$, the bank is in Regime B and $\delta = \delta_B$, so we have proved the third branch of (30).

If (δ, G) satisfies $G \leq \gamma(\delta) D$ and the bank is in Regime A, then by Lemma 3 $X_L = \underline{X} = X_H$. Hence, by (8), $\delta = p\delta_L + (1-p)\delta_H = \delta_e$ and the condition that $G \leq \gamma(\delta) D$ is equivalent to $G \leq \gamma(\delta_e) D$. Altogether $\delta = \delta_e$ if $G \leq \gamma(\delta_e) D$ and we have proven the first branch of (30).

By Lemma 5, $\overline{\delta_B} < \delta_e$ and we know $\partial\gamma/\partial\delta < 0$. Therefore, if $G \in \left(\gamma(\delta_e) D, \gamma\left(\overline{\delta_B}\right) D\right)$, neither is $X_L = \overline{X}$ (i.e. the bank in Regime B) with certainty, nor is $X_L = \underline{X}$ (i.e. the bank in Regime A) with certainty. The bank must play a mixed strategy, which holds true only if $G = \gamma(\delta) D$. It follows that $\delta = \delta_M$ if $G \in \left(\gamma(\delta_e) D, \gamma\left(\overline{\delta_B}\right) D\right)$ and we have proven the second branch of (30). In this branch, $\delta = \gamma^{-1}\left(\frac{G}{D}\right)$ decreases with G because $\partial\gamma/\partial\delta < 0$.

At the threshold $G = \gamma(\delta_e) D$, $\delta_M = \delta_e$ and hence δ is continuous with G. So is δ at the threshold $G = \gamma\left(\overline{\delta_B}\right) D$ where $\delta_M = \overline{\delta_B}$. Therefore, and hence δ is continuous with G throughout.

We have seen δ decreases with G in the second and third branches of (30). Obviously δ is constant in the first and the last branches. Altogether, δ decreases with G throughout. ■

Proof of Lemma 9

Proof. Because $\gamma(R, r_2, \delta) = \omega - (R - (1 - q_L)(1 - \omega) / (\delta - q_L) \times r_2) \delta$, by (66)

$$\omega = G\left(\frac{1}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} \left(R^B\right)^{\frac{1}{1-\alpha}} + \left(R^B - \frac{(1-q_L)(1-\omega)}{\delta - q_L} r^B\left(R^B, \delta, G\right)\right)\delta. \tag{63}$$

The right hand side of (63) increases with R^B because $\partial r^B/\partial R < 0$. It also increases G because $\partial r^B/\partial G < 0$ and with δ because $\partial r^B/\partial\delta < 0$ and condition (58) holds true. The lemma then follows from the Implicit Function theorem. ■

Proof of Lemma 6

Proof. At date 0, $D = D(R)$ and $\gamma = \gamma(R, r_2, \delta) = \omega - (R - (1 - q_L)(1 - \omega) / (\delta - q_L) \times r_2) \delta$ by (24). If the bank is in Regime A, $r_2 = r^A(R, \delta, G)$ by (28). Altogether, the bank chooses $X_L = \underline{X}$ and is in Regime A if

$$\gamma\left(R, r^A(R, \delta, G), \delta\right) D(R) > G. \tag{64}$$

Given that $\partial\gamma/\partial R < 0$, $\partial\gamma/\partial r_2 > 0$ and $\partial r^A/\partial R < 0$; and that $D'(R) < 0$, the left hand side of condition (64) decreases with R and this condition holds if and only if

$$R < R^A(G, \delta),$$

where R^A is the root of the following equation:

$$\gamma\left(R^A, r^A\left(R^A, \delta, G\right), \delta\right) D\left(R^A\right) = G. \tag{65}$$

Similarly, the bank is in Regime B and chooses $X_L = \overline{X}$ if $\gamma D < G$. In Regime B, $r_2 = r^B(R, \delta, G)$, where $r^B(R, \delta, G)$ is defined by (61) and also decreasing with R. Using a similar argument, the bank is in Regime B if and only if

$$R > R^B(G, \delta),$$

where R^B is the root of the following equation:

$$\gamma\left(R^B, r^B\left(R^B, \delta, G\right), \delta\right) D\left(R^B\right) = G. \tag{66}$$

As the left hand side of condition (64), the left hand side of condition (66) decreases with R.

Given that $\partial\gamma/\partial r_2 > 0$ and that $r^A(R,\delta,G) > r^B(R,\delta,G)$, (65) implies that

$$\gamma\left(R^A, r^B\left(R^A,\delta,G\right),\delta\right) D\left(R^A\right) < G,$$

that is, $f\left(R^A\right) < G$, where $f(R) := \gamma\left(R, r^B(R,\delta,G),\delta\right) D(R)$.By definition of R^B, $f\left(R^B\right) = G$. Then $f\left(R^A\right) < f\left(R^B\right)$. We saw above that $f' < 0$. Hence $R^A > R^B$. Moreover, by Lemma 3, $\gamma < \omega$. Therefore, the left hand side of equation (65) is smaller than $\omega D\left(R^A\right)$. From this equation it follows that $G < \omega D\left(R^A\right)$, which is equivalent to $\overline{R}(G) > R^A$. ■

Proof of Lemma 7

Proof. Given that $D = D(R)$, it follows from (33) that

$$V^A = \left\{ \begin{array}{c} D(R) \times \delta_e R + G \text{ if } R \geq \overline{R}(G) \\ D(R) \times \left(\delta_e R - \omega \frac{\delta_e - \delta}{\delta}\right) + \frac{\delta_e}{\delta} G \text{ if } R < \overline{R}(G) \end{array} \right\} \tag{67}$$

and from (34)

$$V^B = \left\{ \begin{array}{c} D(R) \times (p\delta + (1-p)\delta_H) R + G \text{ if } R \geq \overline{R}(G) \\ D(R) \times \left[(p\delta + (1-p)\delta_H) R - \omega \frac{(1-p)(\delta_H - \delta)}{\delta}\right] + \frac{\delta_e + p(\delta - \delta_L)}{\delta} G \text{ if } R < \overline{R}(G) \end{array} \right\}. \tag{68}$$

Compare V^A given by (67) and V^B given by (68). Then the first claim straightforwardly follows from the observation that at $\delta = \delta_L$, $p\delta + (1-p)\delta_H = \delta_e$, $(1-p)(\delta_H - \delta) = \delta_e - \delta$, and $\delta_e + p(\delta - \delta_L) = \delta_e$.

For the second claim, because $R^A < \overline{R}$ by Lemma 6, if $R \in \left[R^B, R^A\right]$, then $R < \overline{R}$ and thus $V^A < V^B$ is equivalent to $\left(R\delta_e - \omega\frac{\delta_e - \delta}{\delta}\right) D(R) + \frac{\delta_e}{\delta} G < D(R)\left[R(p\delta + (1-p)\delta_H) - \omega\frac{(1-p)(\delta_H - \delta)}{\delta}\right] + \frac{\delta_e + p(\delta - \delta_L)}{\delta} G$, which is in turn equivalent to

$$\begin{aligned}
\left[-p(\delta - \delta_L) R + \omega \frac{p(\delta - \delta_L)}{\delta}\right] D(R) &< \frac{p(\delta - \delta_L)}{\delta} G|_{\delta > \delta_L} \Leftrightarrow \\
(-R\delta + \omega) D(R) &< G|_{D(R) = \left(\overline{A}\alpha/R\right)^{\frac{1}{1-\alpha}}} \Leftrightarrow \\
\omega - R\delta &< G\left(\frac{1}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} R^{\frac{1}{1-\alpha}} \Leftrightarrow \\
\omega &< G\left(\frac{1}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} R^{\frac{1}{1-\alpha}} + R\delta. \qquad (69)
\end{aligned}$$

By (63)

$$\omega = G\left(\frac{1}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} \left(R^B\right)^{\frac{1}{1-\alpha}} + \left(R^B - \frac{(1-q_L)(1-\omega)}{\delta - q_L} r^B\left(R^B,\delta,G\right)\right)\delta.$$

It follows that $\omega < G\left(\frac{1}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} \left(R^B\right)^{\frac{1}{1-\alpha}} + R^B\delta$. Hence, inequality (69) holds true for $R = R^B$. Because its right hand side increases with R, it holds true for all $R \in \left[R^B, R^A\right]$. ■

Proof of Lemma 8

Proof. That $c_{B1} = (1 + \omega(1-p)(\delta_H - \delta)/\delta) \times c_{B2} > c_{B2}$ is obvious. That $c_A \geq c_{B1}$ is equivalent to

$$\begin{aligned} \frac{1+\omega(\delta_e - \delta)/\delta}{\delta_e} &\geq \frac{1+\omega(1-p)(\delta_H-\delta)/\delta}{p\delta+(1-p)\delta_H} \Leftrightarrow \\ \frac{p\delta+(1-p)\delta_H}{\delta_e} &\geq \frac{1+\omega(1-p)(\delta_H-\delta)/\delta}{1+\omega(\delta_e-\delta)/\delta}, \end{aligned}$$

which, with both sides subtracting 1 and using fact that $(\delta_e - \delta) = p(\delta_L - \delta) + (1-p)(\delta_H - \delta)$, is equivalent to

$$\frac{p(\delta-\delta_L)}{\delta_e} \geq \frac{p(\delta-\delta_L)\times\omega/\delta}{1+\omega(\delta_e-\delta)/\delta}, \tag{70}$$

which holds with equality if $\delta = \delta_L$. If $\delta > \delta_L$, inequality (70) holds true strictly because it is equivalent to

$$\begin{aligned} \frac{1}{\delta_e} &> \frac{\omega/\delta}{1+\omega(\delta_e-\delta)/\delta} \Leftrightarrow \\ 1+\omega(\delta_e-\delta)/\delta &> \omega\delta_e/\delta \Leftrightarrow \\ 1-\omega+\omega\delta_e/\delta &> \omega\delta_e/\delta. \end{aligned}$$

■

Proof of Lemma 10

Proof. By Lemma 6, $R^B < \overline{R}$. Hence, if $R = R^B$, then $G < \omega D(R)$, or equivalently,

$$G\left(\frac{R}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} - \omega < 0.$$

As a result, $\min\left(G\left(\frac{R}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} - \omega, 0\right) = G\left(\frac{R}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} - \omega$ and $\max\left(G\left(\frac{R}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} - \omega, 0\right) = 0$. Substitute these into (**??**) and observe that in that equation $r_2 = r^B$, and we have

$$\underline{r} = \frac{1}{1-\omega}\frac{\underline{A}}{\overline{A}\alpha}\left(R + \frac{1}{\delta}\left(G\left(\frac{R}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} - \omega\right)\right). \tag{71}$$

At $R = R^B$, $G = \gamma D = \left[\omega - \left(R - (1-q_L)(1-\omega)/(\delta - q_L) \times r^B\right)\delta\right] D(R)$. Since $D(R) = \left(\frac{R}{\overline{A}\alpha}\right)^{\frac{-1}{1-\alpha}}$, it follows that

$$R + \frac{1}{\delta}\left(G\left(\frac{R}{\overline{A}\alpha}\right)^{\frac{1}{1-\alpha}} - \omega\right) = \frac{(1-q_L)(1-\omega)}{\delta - q_L} r^B.$$

Substitute this into (71) and we have

$$\begin{aligned} \underline{r} &= \frac{\underline{A}}{\overline{A}\alpha}\frac{1-q_L}{\delta-q_L} r^B \\ &= \frac{\delta_L - q_L}{\delta - q_L} r^B. \end{aligned}$$

Given that $\delta \geq \delta_L$, $\underline{r_2} \leq r^B$ and

$$\min\left(\underline{r_2}, r^B\right) = \underline{r} = \frac{\delta_L - q_L}{\delta - q_L} r^B. \tag{72}$$

Substitute (72) into (61), and we have

$$r^B\left\{1-(1-p)(1-q_H)+(1-p)(1-q_H)\frac{\delta_L-q_L}{\delta-q_L}\right\}=1,$$

from which

$$r^B=\frac{1}{1-(1-p)(1-q_H)\frac{\delta-\delta_L}{\delta-q_L}}.$$

Given $r_2=r^B$, we find (42).

At $r_2=r^B$,

$$\begin{aligned}\gamma &= \omega-\left(R-\frac{(1-q_L)(1-\omega)}{\delta-q_L}\frac{1}{1-(1-p)(1-q_H)\frac{\delta-\delta_L}{\delta-q_L}}\right)\delta\\ &= \omega-\left(R-\frac{(1-q_L)(1-\omega)}{\delta-q_L-(1-p)(1-q_H)(\delta-\delta_L)}\right)\delta\\ &= \omega-\left(R-\frac{(1-q_L)(1-\omega)}{(1-(1-p)(1-q_H))(\delta-\delta_L)+\delta_L-q_L}\right)\delta.\\ &= \gamma(R,\delta).\end{aligned}$$

To prove $\Gamma(R,\delta)<\omega D(R)$ for any $R>1$, it suffices to prove

$$\frac{(1-q_L)(1-\omega)}{(1-(1-p)(1-q_H))(\delta-\delta_L)+\delta_L-q_L}\le 1,$$

which, because $\delta\ge\delta_L$ and $\delta_L-q_L=(1-q_L)\underline{A}/\left(\overline{A}\alpha\right)$, follows from $1-\omega<\underline{A}/\left(\overline{A}\alpha\right)$, which holds true by Assumption (16).

From fact that $R\le R^B(G,\delta)$ if and only if $G\le\Gamma(R,\delta)$ it follows $G=\Gamma(R,\delta)$ is the inverse function of $R=R^B(G,\delta)$ for any given δ. If $R=R^B(G,\delta)$, then $dR=R_G^{B\prime}dG+R_\delta^{B\prime}d\delta$. By Lemma 9 $R_G^{B\prime}<0$ and $R_\delta^{B\prime}<0$. Hence, $\partial G/\partial R=1/R_G^{B\prime}<0$ and $\partial G/\partial\delta=-R_\delta^{B\prime}/R_G^{B\prime}<0$. ■

Proof of Proposition 4

Proof. For two terms A and B, We define $A\propto B$ if A and B have the same signs. By (37),

$$\Pi'(R)\propto\left\{\begin{array}{c}\frac{1}{\alpha}c_{B2}-R \text{ if } R\ge\overline{R}(G)\\ \frac{1}{\alpha}c_{B1}-R \text{ if } R\in\left[R^B(G,\delta),\overline{R}(G)\right]\\ \frac{1}{\alpha}c_A-R \text{ if } R<R^B(G,\delta)\end{array}\right\}. \tag{73}$$

Consider first the case where $G<\Gamma\left(\frac{1}{\alpha}c_A,\delta\right)$, which is equivalent to $\frac{1}{\alpha}c_A<R^B(G,\delta)$. In this case, if $R<\frac{1}{\alpha}c_A$, then $R<R^B(G,\delta)$ and by (73), $\Pi'(R)=\frac{1}{\alpha}c_A-R>0$. If $R>\frac{1}{\alpha}c_A$, then $R>\max\left(\frac{1}{\alpha}c_A,\frac{1}{\alpha}c_{B1},\frac{1}{\alpha}c_{B2}\right)$ by Lemma 8 and hence $\Pi'(R)<0$ always. Also observe that $\Pi(R)$ is a continuous function by (37). Hence, the point at which $\Pi(R)$ is maximized is $R^*=\frac{1}{\alpha}c_A$.

Second, consider the case where $\Gamma\left(\frac{1}{\alpha}c_A,\delta\right)\le G\le\Gamma\left(\frac{1}{\alpha}c_{B1},\delta\right)$, which is equivalent to $\frac{1}{\alpha}c_A\ge R^B(G,\delta)\ge\frac{1}{\alpha}c_{B1}$. In this case, if $R<R^B(G,\delta)$, then $R<\frac{1}{\alpha}c_A$ and hence $\Pi'(R)\propto\frac{1}{\alpha}c_A-R>0$. If $R>R^B(G,\delta)$, then

$R > \frac{1}{\alpha}c_{B1} = \max\left(\frac{1}{\alpha}c_{B1}, \frac{1}{\alpha}c_{B2}\right)$ by Lemma 8 and hence $\Pi'(R) \propto \frac{1}{\alpha}c_{B1} - R < 0$ or $\Pi'(R) \propto \frac{1}{\alpha}c_{B2} - R < 0$. Hence, $R^* = R^B(G, \delta)$.

Third, consider the case where $G \in \left(\Gamma\left(\frac{1}{\alpha}c_{B1}, \delta\right), \omega D\left(\frac{1}{\alpha}c_{B1}\right)\right)$, or equivalently $R^B(G,\delta) < \frac{1}{\alpha}c_{B1} < \overline{R}(G)$. In this case, if $R < \frac{1}{\alpha}c_{B1}$, then $R < \overline{R}(G)$, then $\Pi'(R) \propto \frac{1}{\alpha}c_{B1} - R$ or $\frac{1}{\alpha}c_A - R$ and is positive in both cases because $R < \frac{1}{\alpha}c_{B1} \leq \frac{1}{\alpha}c_A$. If $R > \frac{1}{\alpha}c_{B1}$, then $R > R^B(G,\delta)$, then $\Pi'(R) \propto \frac{1}{\alpha}c_{B1} - R$ or $\frac{1}{\alpha}c_{B2} - R$, which are both negative because $R > \frac{1}{\alpha}c_{B1} > \frac{1}{\alpha}c_{B2}$. Hence, $R^* = \frac{1}{\alpha}c_{B1}$.

Fourth, consider the case where $G \in \left[\omega D\left(\frac{1}{\alpha}c_{B1}\right), \omega D\left(\frac{1}{\alpha}c_{B2}\right)\right]$, or equivalently $\frac{1}{\alpha}c_{B1} > \overline{R}(G) > \frac{1}{\alpha}c_{B2}$. In this case, if $R < \overline{R}(G)$, then $R < \frac{1}{\alpha}c_{B1}$ and $\Pi'(R) \propto \frac{1}{\alpha}c_{B1} - R$ or $\frac{1}{\alpha}c_A - R$, which are both positive because $R < \frac{1}{\alpha}c_{B1} \leq \frac{1}{\alpha}c_A$. If $R > \overline{R}(G)$, then $R > \frac{1}{\alpha}c_{B2}$ and $\Pi'(R) \propto \frac{1}{\alpha}c_{B2} - R < 0$. Hence, $R^* = \overline{R}(G)$.

Lastly, consider the case where $G > \omega D\left(\frac{1}{\alpha}c_{B2}\right)$ or equivalently $\frac{1}{\alpha}c_{B2} > \overline{R}(G)$. In this case, if $R < \frac{1}{\alpha}c_{B2}$, then $R < \min\left(\frac{1}{\alpha}c_A, \frac{1}{\alpha}c_{B1}, \frac{1}{\alpha}c_{B2}\right)$ and hence $\Pi'(R) > 0$ always. If $R > \frac{1}{\alpha}c_{B2}$, then $R > \overline{R}(G)$ and hence $\Pi'(R) \propto \frac{1}{\alpha}c_{B2} - R < 0$. Hence $R^* = \frac{1}{\alpha}c_{B2}$. ■

Proof of Lemma 11

Proof. $\phi(\delta)$ is defined by the implicit function

$$\delta = \overline{\delta_B}\left(r^B(\delta)/\phi(\delta)\right).$$

By the implicit function theorem, $1 = \overline{\delta_B}' r^{B\prime}\frac{1}{\phi} + \overline{\delta_B}' r^B \frac{-\phi'}{\phi^2} \Leftrightarrow$

$$\phi' = \frac{\phi^2}{\left(-\overline{\delta_B}'\right) r^B}\left(1 + \left(-\overline{\delta_B}'\right) r^{B\prime}\frac{1}{\phi}\right).$$

By Lemma 5, $\overline{\delta_B}' < 0$ and by equation (42) $r^{B\prime} > 0$. Hence, $\phi' > 0$. ■

Proof of Lemma 12

Proof. Let

$$\begin{aligned}
R(\delta) &: = \frac{1}{\alpha}c_A(\delta);\\
x &: = 1 - (1-p)(1-q_H)\\
y &: = (1-q_L)(1-\omega)\\
z &: = (1-x)\delta_L - q_L.
\end{aligned}$$

Then, by Lemma 10, $\Gamma\left(\frac{1}{\alpha}c_A(\delta), \delta\right) = \Gamma(R(\delta), \delta) = D(R(\delta))\left[\omega - \left(R(\delta) - \frac{y}{x\delta+z}\right)\delta\right]$. Hence, $\Gamma'_R = D'\left[\omega - \left(R - \frac{y}{x\delta+z}\right)\delta\right] - D\delta = -D\left[\frac{1}{(1-\alpha)R}\left[\omega - \left(R - \frac{y}{x\delta+z}\right)\delta\right] + \delta\right] = -\frac{D}{1-\alpha}\left[\frac{\omega}{R} + \frac{1}{R}\frac{y}{x\delta+z}\delta - \alpha\delta\right]$ and $\Gamma'_\delta = -D\left[R - \frac{y}{x\delta+z} + \frac{xy\delta}{(x\delta+z)^2}\right] = -D\left[R - \frac{yz}{(x\delta+z)^2}\right]$. By (38),

$$c_A(\delta) = \frac{\omega}{\delta} + \frac{1-\omega}{\delta_e}. \tag{74}$$

Then, $R'(\delta) = -\frac{\omega}{\alpha\delta^2}$. Thus, $\left[\Gamma\left(\frac{1}{\alpha}c_A(\delta), \delta\right)\right]' > 0 \Leftrightarrow$

$$\begin{aligned}
\frac{1}{1-\alpha}\left[\frac{\omega}{R} + \frac{1}{R}\frac{y}{x\delta+z}\delta - \alpha\delta\right]\frac{\omega}{\alpha\delta^2} &> R - \frac{yz}{(x\delta+z)^2} \Leftarrow \\
\frac{1}{1-\alpha}\left[\omega + \frac{y}{x\delta+z}\delta - \alpha R\delta\right]\frac{\omega}{\alpha} &> (R\delta)^2 . \qquad (75)
\end{aligned}$$

Observe that $R\delta = \frac{1}{\alpha}\left(\omega + \frac{\delta}{\delta_e}(1-\omega)\right) < \frac{1}{\alpha}$. Hence, inequality (75) follows from

$$\begin{aligned}
\frac{1}{1-\alpha}\left[\omega + \frac{y}{x\delta+z}\delta - \left(\omega + \frac{\delta}{\delta_e}(1-\omega)\right)\right]\frac{\omega}{\alpha} &> \frac{1}{\alpha^2} \Leftrightarrow \\
\left[\frac{(1-q_L)(1-\omega)}{x\delta+z}\delta - \frac{\delta}{\delta_e}(1-\omega)\right]\omega &> \frac{1-\alpha}{\alpha} \Leftrightarrow \\
\left[\frac{1-q_L}{x\delta+(1-x)\delta_L-q_L} - \frac{1}{\delta_e}\right]\delta\omega(1-\omega) &> \frac{1-\alpha}{\alpha}|_{x<1} \Leftarrow \\
\left[\frac{1-q_L}{\delta-q_L} - \frac{1}{\delta_e}\right]\delta\omega(1-\omega) &> \frac{1-\alpha}{\alpha}. \qquad (76)
\end{aligned}$$

Observe that the left hand side of inequality (76) decreases with δ. Hence, the inequality follows from

$$\begin{aligned}
\left[\frac{1-q_L}{\delta_e-q_L} - \frac{1}{\delta_e}\right]\delta_e\omega(1-\omega) &> \frac{1-\alpha}{\alpha} \Leftrightarrow \\
\frac{q_L(1-\delta_e)}{\delta_e-q_L}\omega(1-\omega) &> \frac{1-\alpha}{\alpha},
\end{aligned}$$

which is inequality (50).

By Lemma 10, $\Gamma\left(\frac{1}{\alpha}c_A(\delta_L), \delta_L\right) > 0 \Leftrightarrow$

$$\omega - \frac{1}{\alpha}c_A(\delta_L)\delta_L + \frac{(1-q_L)(1-\omega)}{\delta_L-q_L}\delta_L > 0. \qquad (77)$$

By (74) $c_A(\delta_L)\delta_L = \left(\frac{\omega}{\delta_L} + \frac{1-\omega}{\delta_e}\right)\delta_L < 1$. Hence, inequality (77) follows from

$$\begin{aligned}
\omega + \frac{(1-q_L)(1-\omega)}{\delta_L-q_L}\delta_L &> \frac{1}{\alpha} \Leftrightarrow \\
(1-\omega)\left(\frac{(1-q_L)}{\delta_L-q_L}\delta_L - 1\right) &> \frac{1-\alpha}{\alpha} \Leftrightarrow \\
(1-\omega)\frac{q_L(1-\delta_L)}{\delta_L-q_L} &> \frac{1-\alpha}{\alpha}. \qquad (78)
\end{aligned}$$

Observe that $\frac{1-x}{x-q_L}$ is a decreasing function over $x \in (q_L, 1)$. Therefore, inequality (78) follows from assumption (50) because $(1-\omega)\frac{q_L(1-\delta_L)}{\delta_L-q_L} > (1-\omega)\frac{q_L(1-\delta_e)}{\delta_e-q_L} > (1-\omega)\omega\frac{q_L(1-\delta_e)}{\delta_e-q_L} > \frac{1-\alpha}{\alpha}$.

Lastly, as $R^{NW} = \frac{1}{\alpha}c_A(\delta_e)$, $R^{NW} < \frac{1}{\alpha}c_{B1}\left(\underline{\delta_2}\right)$ if and only $c_A(\delta_e) < c_{B1}\left(\underline{\delta_2}\right)$, which, by (74) and (39), is equivalent to $\frac{1}{\delta_e} < \frac{1+\omega(1-p)\left(\delta_H-\underline{\delta_2}\right)/\underline{\delta_2}}{p\underline{\delta_2}+(1-p)\delta_H} \Leftrightarrow \frac{p\underline{\delta_2}+(1-p)\delta_H}{\delta_e} < 1 + \frac{\omega(1-p)\left(\delta_H-\underline{\delta_2}\right)}{\underline{\delta_2}}$, which, using equation $\delta_e = p\delta_L + (1-p)\delta_H$ and certain rearrangements, is equivalent to

$$\frac{p\left(\underline{\delta_2}-\delta_L\right)}{(1-p)\left(\delta_H-\underline{\delta_2}\right)}\frac{\underline{\delta_2}}{\delta_e} < \omega. \qquad (79)$$

$\underline{\delta_2}$ satisfies $\phi\left(\underline{\delta_2}\right) = \frac{1}{\alpha} c_{B1}\left(\underline{\delta_2}\right)$, where $\phi(\cdot)$ is defined in equation (46). It follows that

$$\frac{p\left(\underline{\delta_2} - \delta_L\right)}{(1-p)\left(\delta_H - \underline{\delta_2}\right)} = 1 - \frac{\alpha}{c_{B1}\left(\underline{\delta_2}\right)} \frac{(1-q_L)(1-\omega)}{(1-x)\underline{\delta_2} + x\delta_L - q_L}. \tag{80}$$

where $x = (1-p)(1-q_H)$. Substitute equation (80) into inequality (79), and the inequality is equivalent to

$$\mu(1 - \lambda(1-\omega)) < \omega, \tag{81}$$

where $\mu := \frac{\underline{\delta_2}}{\delta_e} < 1$ and

$$\lambda := \frac{\alpha}{c_{B1}\left(\underline{\delta_2}\right)} \frac{1-q_L}{(1-x)\underline{\delta_2} + x\delta_L - q_L}. \tag{82}$$

Observe, first, that inequality (81) holds for any $\omega \in (0,1)$ if $\lambda > 1$, because either $1 - \lambda(1-\omega) \leq 0$ and then inequality (81) holds, or $1 - \lambda(1-\omega) > 0$ in which case the left hand side of (81) is smaller than $1 - \lambda(1-\omega)$ (as $\mu < 1$) and $1 - \lambda(1-\omega) < \omega \Leftrightarrow 1 - \omega < \lambda(1-\omega)$, which holds true if $\lambda > 1$. Therefore, to prove $R^{NW} < \frac{1}{\alpha} c_{B1}\left(\underline{\delta_2}\right)$, it suffices to prove that $\lambda > 1$ under assumption (50). With λ defined in (82) and $c_{B1}(\delta)$ given by (39), $\lambda > 1 \Leftrightarrow$

$$\begin{aligned}
\frac{1-q_L}{(1-x)\underline{\delta_2} + x\delta_L - q_L} &> \frac{1}{\alpha} \frac{1 + \omega(1-p)\left(\delta_H - \underline{\delta_2}\right)/\underline{\delta_2}}{p\underline{\delta_2} + (1-p)\delta_H} \Leftarrow \\
\frac{1-q_L}{(1-x)\underline{\delta_2} + x\delta_L - q_L} &> \frac{1}{\alpha} \frac{1 + (1-p)\left(\delta_H - \underline{\delta_2}\right)/\underline{\delta_2}}{p\underline{\delta_2} + (1-p)\delta_H} \Leftrightarrow \\
\frac{(1-q_L)\underline{\delta_2}}{(1-x)\underline{\delta_2} + x\delta_L - q_L} &> \frac{1}{\alpha}.
\end{aligned} \tag{83}$$

Under assumption (50),

$$\frac{1-\alpha}{\alpha} < \frac{q_L(1-\delta_e)}{\delta_e - q_L}.$$

It follows that

$$\frac{1}{\alpha} < \frac{\delta_e(1-q_L)}{\delta_e - q_L}.$$

Using this inequality, inequality (83) follows from

$$\begin{aligned}
\frac{(1-q_L)\underline{\delta_2}}{(1-x)\underline{\delta_2} + x\delta_L - q_L} &> \frac{\delta_e(1-q_L)}{\delta_e - q_L} \Leftrightarrow \\
\frac{\underline{\delta_2}}{(1-x)\underline{\delta_2} + x\delta_L - q_L} &> \frac{\delta_e}{\delta_e - q_L},
\end{aligned}$$

which holds true because $\frac{\underline{\delta_2}}{(1-x)\underline{\delta_2} + x\delta_L - q_L} > \frac{\underline{\delta_2}}{\underline{\delta_2} - q_L} > \frac{\delta_e}{\delta_e - q_L}$, where the first inequality is due to the fact that $\underline{\delta_2} > \delta_L$ and the second the facts that $\underline{\delta_2} < \delta_e$; and that $\frac{x}{x - q_L}$ decreases with x for $x > q_L$. ∎

Proof of Lemma 13

Proof. We prove that There exists a $\delta_3^* \in [\delta_L, \delta_H)$ such that $\varphi' > 0$ for $\delta < \delta_3^*$ and $\varphi' < 0$ for $\delta > \delta_3^*$. Let

$$\begin{aligned} R &= \frac{1}{\alpha} c_{B1}(\delta); \\ \lambda(\delta) \quad : &= \frac{\omega(1-p)\frac{\delta_H}{\delta}}{(1-\omega(1-p))\delta + \omega(1-p)\delta_H} + \frac{p}{p\delta + (1-p)\delta_H}. \end{aligned}$$

Then $\varphi' < 0 \Leftrightarrow \left[\frac{\omega}{(1-\alpha)R} - \frac{\alpha}{1-\alpha}\frac{p(\delta-\delta_L)\delta}{(1-p)(\delta_H-\delta)}\right]\lambda(\delta) < \frac{p}{1-p}\left[\frac{\delta-\delta_L}{\delta_H-\delta} + \frac{(\delta_H-\delta_L)\delta}{(\delta_H-\delta)^2}\right] \Leftrightarrow$

$$\frac{\omega}{1-\alpha} < \frac{p}{1-p} R\delta \left\{ \frac{1}{\lambda(\delta)} \left[\frac{1-\frac{\delta_L}{\delta}}{\delta_H - \delta} + \frac{\delta_H - \delta_L}{(\delta_H-\delta)^2} \right] + \frac{\alpha}{1-\alpha}\frac{\delta-\delta_L}{\delta_H-\delta} \right\}. \tag{84}$$

Observe that $R\delta = \frac{1}{\alpha}\frac{(1-\omega(1-p))\delta+\omega(1-p)\delta_H}{p\delta+(1-p)\delta_H} = \frac{1}{\alpha}\frac{1-\omega(1-p)}{p}\left(1 - \frac{(1-p)/p-\omega(1-p)/(1-\omega(1-p))}{\delta+(1-p)\delta_H/p}\delta_H\right)$ increases with δ; and that $\lambda(\delta)$ is an decreasing function, hence $\frac{1}{\lambda(\delta)}$ increasing. Therefore, the right hand side of inequality (84) increases with δ. It follows that there exists δ_3^* such that inequality (84) holds – namely $\varphi' < 0$ – if $\delta > \delta_3^*$ and the inverse inequality holds – namely $\varphi' < 0$ – if $\delta < \delta_3^*$. Hence, we prove the first claim of the lemma.

$\delta_3^* = \delta_L$, that is, $\varphi' < 0$ for all $\delta \in (\delta_L, \delta_H)$, if inequality (84) weakly holds at $\delta = \delta_L$, which is equivalent to $\frac{\omega}{1-\alpha}\lambda(\delta_L) \le \frac{p}{1-p}\frac{1}{\alpha}c_{B1}(\delta_L)\delta_L\frac{1}{\delta_H-\delta_L}$, which, because $c_{B1}(\delta_L)\delta_L = \frac{\delta_L+\omega(1-p)(\delta_H-\delta_L)}{\delta_e}$, is in turn equivalent to

$$\frac{\alpha}{1-\alpha}\frac{\omega}{\delta_L + \omega(1-p)(\delta_H-\delta_L)}\lambda(\delta_L) \le \frac{p}{1-p}\frac{1}{\delta_e(\delta_H-\delta_L)}, \tag{85}$$

that is, $\alpha/(1-\alpha)$ is below a threshold. To find a sufficient condition for (85), observe that $\frac{\omega}{\delta_L+\omega(1-p)(\delta_H-\delta_L)}$ increases with ω and hence do does $\lambda(\delta_L)$. Therefore, inequality (85) holds true for any $\omega < 1$ if it holds true at $\omega = 1$, that is, if

$$\frac{\alpha}{1-\alpha}\frac{\delta_H-\delta_L}{\delta_L} \le \frac{p}{1-p},$$

that is, condition (56). ■

8 Appendix B: The other case of Phase 3

If $1/(1-\alpha) > 1/(1-\alpha_3)$, then by Lemma 13, the maximum point $\delta_3^* > \delta_L$ and the graph of $\varphi(\delta)$ is in a "$\wedge$" shape. In this case, there are two equilibria if $G \in \left[\omega D\left(\frac{1}{\alpha}c_{B1}(\delta_L)\right), \varphi(\delta_3^*)\right)$, of which the discount factors move in opposite directions with a rise in G, as is illustrated in the figure below.

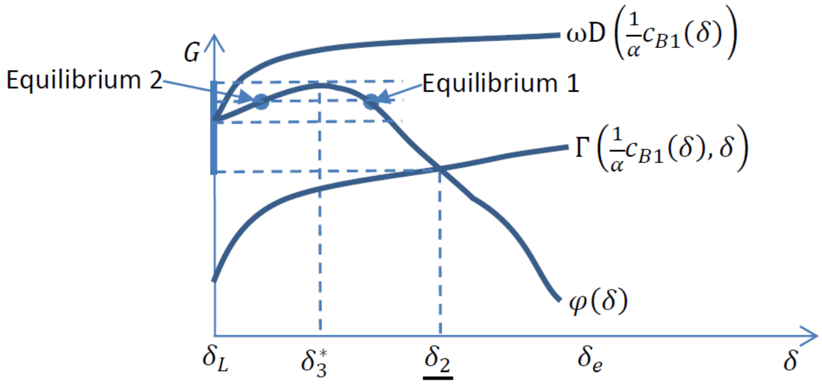

Figure B1: The case of Phase 3 in which $\delta_3^* > \delta_L$. If $G \in \left[\omega D\left(\frac{1}{\alpha}c_{B1}(\delta_L)\right), \varphi(\delta_3^*)\right)$, there are two equilibria: Equilibrium 1 and Equilibrium 2. The former sees δ decreasing with G, the latter increasing.

The equilibrium multiplicity is driven by the complementarity between the optimal lending rate R^* and the discount factor δ. At a given G, if at date 0 the bank expects δ to be high, then it judges the lending cost is low and thus charges a low lending rate (as $R^* = \frac{1}{\alpha}c_{B1}(\delta)$ is decreasing). As a result, the lending scale $D(R)$ is large and so is the genuine liquidity need $\omega D - G$ that is met with borrowing, which induce the investors to discount the loans with a high δ indeed (as $\delta'(R) < 0$ by equation 31, taking into account $D = D(R)$). By a parallel argument, if at date 0 the bank expects a low δ, this expectation can be self-fulfilling too if the complementarity is strong. The strength of the complementarity increases with the scale of $D'(R)$, which is in proportional to $1/(1-\alpha)$. Therefore, the equilibrium multiplicity occurs if and only if $1/(1-\alpha)$ is large enough.

Equilibrium 2 in Figure B1 above, in which the discount factor increases with G, is unstable in the sense of Malherbe (2014). To show that, following Malherbe (2014), we construct a mapping from the (rational) expectation of the discount factor δ' that the bank holds at date 0 to the discount factor δ that investors use at date 1. In Phase 3, at date 0 the bank chooses a lending rate of $R(\delta') = \frac{1}{\alpha}c_{B1}(\delta')$ and lending scale of $D = \left(\overline{A}\alpha\right)^{\frac{1}{1-\alpha}} R(\delta')^{\frac{-1}{1-\alpha}}$. Substitute these into equation 31, we find the mapping $\delta = f(\delta')$ is determined by

$$\frac{\delta-\delta_L}{\delta_H-\delta}\delta = \frac{1-p}{p}\left(\frac{\omega}{R(\delta')} - \frac{G}{\left(\overline{A}\alpha\right)^{\frac{1}{1-\alpha}}}R(\delta')^{\frac{\alpha}{1-\alpha}}\right). \tag{86}$$

The mapping has the following properties. First, $f' > 0$. That is because the left hand side of (86) is increasing function of δ, while the right hand side is an decreasing function of R, and, as $R'(\delta') < 0$, an increasing function of δ'. Second, if $G > \omega D\left(\frac{1}{\alpha}c_{B1}(\delta_L)\right)$, then $f(\delta_L) < \delta_L$. That is because the right hand side of (86) has the same sign as $\omega D(R(\delta')) - G$ and hence is negative at $\delta' = \delta_L$ if $G > \omega D\left(\frac{1}{\alpha}c_{B1}(\delta_L)\right)$. Third, $f(\delta_H) < \delta_H$. That is because at $\delta' = \delta_H$, the value on the the right hand side of (86) is finite. An equilibrium is a fixed point of mapping f: From Figure 1B above, if $G \in \left[\omega D\left(\frac{1}{\alpha}c_{B1}\left(\delta_L\right)\right), \varphi\left(\delta_3^*\right)\right)$, the mapping has two fixed points and can hence be illustrated as follows.

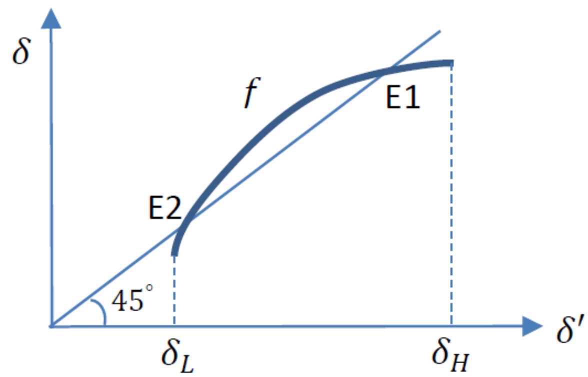

Figure B2: When there are two equilibrium, the one in which δ has a smaller value is unstable.

It is straightforward to see that the Equilibrium 2 of Figure 1B, in which δ has a smaller value, is unstable, as at that point $f' > 1$, while the Equilibrium 1 is stable.